Central leaves and EKOR strata on Shimura varieties with parahoric reduction

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1. Gutachten: Prof. Dr. Torsten Wedhorn
2. Gutachten: Prof. Dr. Timo Richarz
Darmstadt – D 17

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Zusammenfassung

Wir untersuchen die Geometrie der speziellen Faser des ganzzahligen Modells einer Shimura-Varietät mit parahorischem Level bei einer gegebenen Primstelle. In den Fällen, die wir betrachten, hängt diese Varietät eng mit einem Modulproblem zusammen und aus diesem Zusammenhang ergeben sich mehrere Möglichkeiten, die Varietät zu unterteilen.

Genauer behandeln wir in dieser Situation zum einen die Definition der zentralen Blätter, deren lokale Abgeschlossenheit und die Beziehung zwischen den Foliationen bei sich änderndem parahorischen Level. Dies steht in Zusammenhang mit der Verifikation von Axiomen für ganzzahlige Modelle, die von He und Rapoport formuliert wurden. Indem wir die Igusa-Varietäten und die Fast-Produkt-Struktur untersuchen, zeigen wir insbesondere, dass man surjektive Abbildungen zwischen den zentralen Blättern für verschiedene Levelstrukturen erhält.

Zum anderen behandeln wir die EKOR-Stratifizierung, die zwischen der Ekedahl-Oort- und der Kottwitz-Rapoport-Stratifizierung interpoliert. Im Siegel-Fall geben wir eine geometrische Beschreibung, indem wir die Theorie der G-Zips von Moonen, Wedhorn, Pink und Ziegler für unseren Kontext geeignet verallgemeinern. Dies gelingt im Wesentlichen, indem wir Gitter durch Gitterketten ersetzen und unseren Blick auf einen gewissen zulässigen Ort verengen.

Contents

Introduction

Shimura varieties are objects of arithmetic geometry (namely varieties over number fields) that naturally arise in the search for generalized, non-abelian reciprocity laws (i.e., in the Langlands program) and as moduli spaces of abelian varieties (with certain extra structures on them). One way of approaching these objects is to try to understand their mod-p reduction (which has to be carefully defined first). Insofar as a moduli interpretation in the above sense exists and continues to exist likewise for the mod-p reduction[1], it allows us to stratify the moduli space according to several invariants of the abelian varieties parametrized, e.g., the isomorphism classes of their p-torsion. (An important observation is that these stratifications genuinely live in the characteristic p world, making use of Frobenius endomorphisms and so on.) This, very roughly, is the general theme everything in this thesis revolves around.

More precisely, we will be dealing with Shimura varieties of Hodge type and parahoric level structure, at some fixed prime $v \mid p$ of the number field over which the Shimura variety is defined. Under some reasonably mild assumptions, cf. 1.18, Kisin and Pappas [KP15] constructed a canonical integral model for such a Shimura variety. We try to understand some aspects of the geometry of the special fiber of said integral model, namely:

(Chapter 2) the central leaves (roughly the patches where the isomorphism class of the p-divisible group associated with the abelian variety is constant); their being locally closed and how they vary as the (parahoric) level is varied.

(Chapter 3) the EKOR strata (an interpolation between the Ekedahl-Oort strata, which in the case of hyperspecial level are roughly the patches where the isomorphism class of the p-torsion associated with the abelian variety is constant, and the Kottwitz-Rapoport strata, which roughly are the patches where the Hodge filtration looks constant) and defining them in a geometrical way.

[1] There need not be a *literal* moduli interpretation, but in any event the stratifications in question derive from a close connection to moduli problems.

Let us now go into more detail, starting with Chapter 2.

On the integral model $\mathscr{S}_K$ (K parahoric level) we have a "universal" abelian scheme (the quotation marks indicating that it is not really universal for some moduli problem on $\mathscr{S}_K$, but it comes from a universal abelian scheme via pullback) and we have various kinds of Hodge tensors. We also have a "universal" isogeny chain of abelian schemes tightly connected to the "universal" abelian scheme. We define the *naive central leaves* on the special fiber $\overline{\mathscr{S}}_K$ to be the loci where the isomorphism type of the geometric fibers of the p-divisible group associated with the "universal" abelian scheme (the "universal" p-divisible group) is constant (alternatively, we may use the "universal" isogeny chain). We arrive at the non-naive version by taking into account the Hodge tensors. This is the content of section 2.1.

Next we show

Theorem A (Corollary 2.12) *The central leaves are locally closed.*

using a somewhat simpler construction than the one given in [HK17]. By foundational work of Oort [Oor04] we already know the naive central leaves to be locally closed. We show that the central leaves are open and closed inside the naive central leaves. Some basic topological considerations allow us to treat this question in perfected formal neighborhoods, allowing us to phrase it as a question about p-divisible groups with crystalline Tate tensors; a question which was answered by Hamacher [Ham17]. This forms section 2.2.

Then we consider (tensor-respecting) self-quasi-isogenies of p-divisible groups (with tensors). The main take-away here is that, if we consider the geometric fibers of the "universal" isogeny chain of p-divisible groups in the above setting, then

Theorem B (Example 2.21) *The self-quasi-isogenies are independent of the level.*

In section 2.4 we recall the almost product structure (under an additional technical assumption 2.23 concerning the Rapoport-Zink uniformization map, which is satisfied e.g. if our reductive group over $\mathbb{Q}_p$ is residually split[2] [Zho18]), which expresses a variant of the Igusa variety (the *Newton-Igusa variety*, with quasi-isogenies instead of isomorphisms) as a product of the Igusa variety and the Rapoport-Zink space.

We apply this to the question of how the central leaves behave under change of the parahoric level. We begin by constructing the change-of-parahoric morphisms on the

[2] A reductive group $G/\mathbb{Q}_p$ is *residually split* if it has the same rank as its base change to the maximal unramified extension of $\mathbb{Q}_p$. The name "residually split" derives from the fact that in this case the maximal reductive quotients of parahoric group schemes associated with G are split reductive.

Shimura varieties, Igusa varieties, Rapoport-Zink spaces and Newton-Igusa varieties. The almost product isomorphism is compatible with these morphisms (Lemma 2.38).

From this we can derive

Theorem C (Corollary 2.42) *The map between Igusa varieties for varying parahoric level is an isomorphism.*

This implies in particular

Theorem D (Corollary 2.43) *The change-of-parahoric map between central leaves is surjective.*

This is the most difficult part of the axioms on integral models He and Rapoport give in [HR17]. Our proof is independent of the one by Rong Zhou in [Zho18], even though we use some of the results he gave in the first version of the cited preprint, which did not contain a proof of the surjectivity. Positively answering another conjecture by He and Rapoport [HR17, Rmk. 3.4], we show

Theorem E (Corollary 2.48) *The change-of-parahoric map between central leaves is the composition of a flat universal homeomorphism of finite type and a finite étale morphism.*

In Chapter 3, the overarching goal (and what we meant above by "defining the EKOR strata in a geometrical way") is to construct a "nice" algebraic stack $\overline{\mathcal{G}}_K\text{-}\mathrm{EKORZip}$ and a "nice" morphism $\overline{\mathscr{S}}_K \to \overline{\mathcal{G}}_K\text{-}\mathrm{EKORZip}$ from the mod-p reduction of the Shimura variety to it, such that the fibers are the EKOR strata. Shen, Yu and Zhang [SYZ19] solved this problem on individual Kottwitz-Rapoport strata and globally after perfection, but not in the form stated here (i.e., globally without passing to perfections). In the Siegel case we propose a solution which specializes to that of Shen, Yu and Zhang on Kottwitz-Rapoport strata, and should not be difficult to generalize to many (P)EL cases. We show that $\overline{\mathscr{S}}_K \to \overline{\mathcal{G}}_K\text{-}\mathrm{EKORZip}$ is surjective. However, we have to leave the question of whether $\overline{\mathscr{S}}_K \to \overline{\mathcal{G}}_K\text{-}\mathrm{EKORZip}$ is smooth (which would be part of "nice") an open conjecture.

For hyperspecial level, the EKOR stratification agrees with the Ekedahl-Oort stratification, and the goal just set out is achieved by the stack of $\overline{\mathcal{G}}_K$-zips, first defined in special cases by Moonen and Wedhorn in [MW04] and then generally by Pink, Wedhorn and Ziegler in [PWZ11; PWZ15]; the relation to Shimura varieties being established in increasing generality in [MW04], by Viehmann and Wedhorn in [VW13], and finally by Zhang in [Zha15].

One way of looking at the transition from hyperspecial level to general parahoric level (at the very least in nice enough (P)EL cases) is from the point of view of moduli problems of abelian varieties with extra structure, where in the hyperspecial case we are really dealing just with that and in the general case we are dealing with isogeny chains of abelian varieties with extra structure, indexed by lattice chains coming from the Bruhat-Tits building of the reductive p-adic Lie group in question. The basic idea in generalizing zips from the hyperspecial to the general parahoric case then is that one should be dealing with chains of zips in the old sense.

The zip of an abelian variety encodes the following information: the Hodge filtration, the conjugate filtration, and the Cartier isomorphism relating the two. In the general case, every abelian variety in the isogeny chain has a Hodge filtration, a conjugate filtration and a Cartier isomorphism. Problems now arise because we are dealing with p-primary isogenies on p-torsion points, implying that the transition morphisms in these chains have non-vanishing kernels. This introduces additional difficulty compared to the hyperspecial case; there is a naive way of defining a zip stack, but eventually we need to consider a certain admissible locus in it, which so far suffers from the absence of a nice moduli description. Passing to perfections however simplifies things and allows us to prove that the admissible locus is closed. From here we arrive at the stack that we are really interested in by dividing out a certain group action involving the unipotent radical of the special fiber of the parahoric group scheme. A careful inspection shows that on Kottwitz-Rapoport strata we arrive at the same result as in [SYZ19].

To sum up the results of Chapter 3,

Theorem F *In the Siegel case, there is an algebraic stack $\overline{\mathcal{G}}_K\text{-EKORZip}$ and a surjective morphism $\overline{\mathscr{S}}_K \to \overline{\mathcal{G}}_K\text{-EKORZip}$, whose fibers are the EKOR strata and such that on Kottwitz-Rapoport strata, one gets the stack and map constructed in [SYZ19].*

For $\mathrm{GSp}(4)$ we do some calculations to illustrate the theory; section 3.2.5.

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1 Background

1.1 Shimura data of Hodge type

This thesis deals with aspects of the geometry of Shimura varieties (of Hodge type), which are the (systems of) varieties associated with Shimura data (of Hodge type).

Definition 1.1. A *Shimura datum of Hodge type* is a pair (G, X), where G is a reductive algebraic group over $\mathbb{Q}$ and $X \subseteq \mathrm{Hom}_{\mathbb{R}\text{-grp}}(\mathbb{S}, G_{\mathbb{R}})$ is a $G(\mathbb{R})$-conjugacy class ($\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}$ being the Deligne torus) subject to the following conditions:

(1) For $h \in X$, the induced Hodge structure $\mathbb{S} \xrightarrow{h} G_{\mathbb{R}} \xrightarrow{\mathrm{Ad}} \mathrm{GL}(\mathrm{Lie}(G_{\mathbb{R}}))$ satisfies $\mathrm{Lie}(G_{\mathbb{C}}) = \mathrm{Lie}(G_{\mathbb{C}})^{-1,1} \oplus \mathrm{Lie}(G_{\mathbb{C}})^{0,0} \oplus \mathrm{Lie}(G_{\mathbb{C}})^{1,-1}$.

(2) $\mathrm{int}(h(i)) \colon G^{\mathrm{ad}}_{\mathbb{R}} \to G^{\mathrm{ad}}_{\mathbb{R}}$ is a Cartan involution, i.e., $\{g \in G^{\mathrm{ad}}(\mathbb{C}) \mid gh(i) = h(i)\overline{g}\}$ is compact. Another way of phrasing this condition: Every finite-dimensional real representation V of $G^{\mathrm{ad}}_{\mathbb{R}}$ carries a $G^{\mathrm{ad}}_{\mathbb{R}}$-invariant bilinear form φ such that $(u, v) \mapsto \varphi(u, h(i)v)$ is symmetric and positive definite. It is enough to show that this holds for one *faithful* finite-dimensional real representation V.

(3) G^{ad} *cannot* be non-trivially written as $G^{\mathrm{ad}} \cong H \times I$ over $\mathbb{Q}$ with $\mathbb{S} \to G_{\mathbb{R}} \xrightarrow{\mathrm{proj}} H_{\mathbb{R}}$ trivial.

(4) There exists an embedding $(G, X) \hookrightarrow (\mathrm{GSp}(V), S^{\pm})$, where $(\mathrm{GSp}(V), S^{\pm})$ is the Shimura datum associated with a finite-dimensional symplectic $\mathbb{Q}$-vector space V (see below). That is, we have an embedding $G \hookrightarrow \mathrm{GSp}(V)$ of $\mathbb{Q}$-group schemes such that the induced map $\mathrm{Hom}_{\mathbb{R}\text{-grp}}(\mathbb{S}, G_{\mathbb{R}}) \hookrightarrow \mathrm{Hom}_{\mathbb{R}\text{-grp}}(\mathbb{S}, \mathrm{GSp}(V_{\mathbb{R}}))$ restricts to a map $X \hookrightarrow S^{\pm}$.

Example 1.2. Let W be a finite-dimensional $\mathbb{R}$-vector space.

$\mathbb{R}$-group homomorphisms $\mathbb{S} \to \mathrm{GL}(W)$ then correspond to Hodge decompositions of W, i.e., to decompositions $W_{\mathbb{C}} = \oplus_{(p,q) \in \mathbb{Z}^2} W^{p,q}_{\mathbb{C}}$, such that $W^{p,q}_{\mathbb{C}}$ is the complex conjugate of $W_{\mathbb{C}}^{q,p}$ for all $(p,q) \in \mathbb{Z}^2$. Under this correspondence, $h\colon \mathbb{S} \to \mathrm{GL}(W)$ corresponds to the Hodge decomposition $W_{\mathbb{C}}^{p,q} = \{w \in W_{\mathbb{C}} \mid \forall z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times : h(z)w = z^{-p}\bar{z}^{-q}w\}$. Hodge decompositions of W of type $(-1,0)+(0,-1)$ correspond to complex structures on W: If $h\colon \mathbb{S} \to \mathrm{GL}(W)$ yields such a Hodge decomposition, then $h(i)$ gives an $\mathbb{R}$-endomorphism J of W with $J \circ J = -\mathrm{id}_W$.

Let $V = (V, \psi)$ be a finite-dimensional symplectic $\mathbb{Q}$-vector space. We say that a complex structure J on $V_{\mathbb{R}}$ is positive (resp. negative) if $\psi_J := \psi_{\mathbb{R}}(\_, J\_)$ is a positive definite (resp. negative definite) symmetric bilinear form on $V_{\mathbb{R}}$. Define S^+ (resp. S^-) to be the set of positive (resp. negative) complex structures on $(V_{\mathbb{R}}, \psi_{\mathbb{R}})$ and $S^\pm := S^+ \sqcup S^-$.

We can make this more concrete: A symplectic basis of $(V_{\mathbb{R}}, \psi_{\mathbb{R}})$ is a basis $e_1, \dots, e_g, e_{-g}, \dots, e_{-1}$, such that $\psi_{\mathbb{R}}$ is of the form

$$\begin{pmatrix} & \tilde{I}_g \\ -\tilde{I}_g & \end{pmatrix}$$

with respect to this basis, where $\tilde{I}_g = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$ is the antidiagonal identity matrix.[1]

Let J be the endomorphism of $V_{\mathbb{R}}$ of the form

$$\begin{pmatrix} & -\tilde{I}_g \\ \tilde{I}_g & \end{pmatrix}$$

with respect to this basis. Then $J \in S^+$ and what we have described is a surjective map

$$\{\text{symplectic bases of } (V_{\mathbb{R}}, \psi_{\mathbb{R}})\} \twoheadrightarrow S^+.$$

In particular we see that $\mathrm{Sp}(V_{\mathbb{R}}, \psi_{\mathbb{R}}) := \{f \in \mathrm{GL}(V_{\mathbb{R}}) \mid \psi_{\mathbb{R}}(f(\_), f(\_)) = \psi_{\mathbb{R}}\}$ (by virtue of acting simply transitively on the symplectic bases) acts transitively on $S^+ \cong \mathrm{Sp}(V_{\mathbb{R}}, \psi_{\mathbb{R}})/\mathrm{SpO}(V_{\mathbb{R}}, \psi_{\mathbb{R}}, J)$ (where we define $\mathrm{SpO}(V_{\mathbb{R}}, \psi_{\mathbb{R}}, J) := \mathrm{Sp}(V_{\mathbb{R}}, \psi_{\mathbb{R}}) \cap O(V_{\mathbb{R}}, \psi_J) = U((V_{\mathbb{R}}, J), \psi_J)$ for a fixed choice of $J \in S^+$) and therefore the general symplectic group $\mathrm{GSp}(V_{\mathbb{R}}, \psi_{\mathbb{R}}) := \{f \in \mathrm{GL}(V_{\mathbb{R}}) \mid \psi_{\mathbb{R}}(f(\_), f(\_)) = c \cdot \psi_{\mathbb{R}} \text{ for some } c \in \mathbb{R}^\times\}$ acts transitively on $S^\pm$ (note that the element of the form $e_{\pm i} \mapsto e_{\mp i}$ of $\mathrm{GSp}(V_{\mathbb{R}}, \psi_{\mathbb{R}})$ for any given choice of symplectic basis $(e_i)_i$ permutes S^+ and S^-).

[1] Occasionally (in particular when doing concrete matrix calculations), it is more convenient to number the basis vectors $1, \dots, g, -1, \dots, -g$ instead of $1, \dots, g, -g, \dots, -1$. Then the standard symplectic form is given by $\begin{pmatrix} & I_g \\ -I_g & \end{pmatrix}$, I_g being the $g \times g$ identity matrix.

Definition 1.3. Condition (1) of Definition 1.1 implies that the action of $\mathbb{G}_{m,\mathbb{R}}$ (embedded in $\mathbb{S}$ in the natural way) on $\mathrm{Lie}(G_{\mathbb{R}})$ is trivial, so that h induces a homomorphism $w\colon \mathbb{G}_{m,\mathbb{R}} \to \mathrm{Cent}(G_{\mathbb{R}})$. This homomorphism is independent of the choice of $h \in X$ and is called the *weight homomorphism* of (G, X).

Moreover, we denote by $\{\mu\}$ the the $G(\mathbb{C})$-conjugacy class of the cocharacter $\mu_h := h \circ (\mathrm{id}_{\mathbb{G}_{m,\mathbb{C}}}, 1)\colon \mathbb{G}_{m,\mathbb{C}} \to \mathbb{G}_{m,\mathbb{C}}^2 \cong \mathbb{S}_{\mathbb{C}} \to G_{\mathbb{C}}$, where h is as above. Obviously, the conjugacy class $\{\mu\}$ is independent of the particular choice of $h \in X$.

Remark 1.4. Let $L/\mathbb{Q}$ be a field extension such that G_L contains a split maximal torus T. Let $W := \mathrm{Norm}_{G(L)}(T)/T$ be the Weyl group. Then the natural map

$$W\backslash \mathrm{Hom}_{L\text{-grp}}(\mathbb{G}_{m,L}, T) \to G(L)\backslash \mathrm{Hom}_{L\text{-grp}}(\mathbb{G}_{m,L}, G_L)$$

is bijective.

Since the left hand side remains unchanged if we go from $L = \bar{\mathbb{Q}}$ (where as usual $\bar{\mathbb{Q}}$ denotes an algebraic closure of $\mathbb{Q}$) to $L = \mathbb{C}$, we see that $\{\mu\}$ contains a cocharacter defined over $\bar{\mathbb{Q}}$ and that we may then also consider $\{\mu\}$ as a $G(\bar{\mathbb{Q}})$-conjugacy class.

Definition 1.5. The *reflex field* $\mathbf{E} = \mathbf{E}(G, X)$ of (G, X) is the field of definition of $\{\mu\}$, i.e., the fixed field in $\bar{\mathbb{Q}}$ of $\{\gamma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \mid \gamma(\{\mu\}) = \{\mu\}\}$.

Example 1.6. The reflex field of the Shimura datum $(\mathrm{GSp}_{2g,\mathbb{Q}}, S^{\pm})$ of Example 1.2 is $\mathbb{Q}$. To wit, one of the cocharacters in the conjugacy class $\{\mu\}$ is

$$\mu(z) = \begin{pmatrix} z & & & & & \\ & \ddots & & & & \\ & & z & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}.$$

Notation 1.7. We denote the ring of (rational) adeles by $\mathbb{A} := \mathbb{A}_{\mathbb{Q}}$, the subring of finite adeles by $\mathbb{A}_f := \mathbb{A}_{\mathbb{Q},f}$ and the subring of finite adeles away from some fixed prime p by $\mathbb{A}_f^p$.

Definition and Remark 1.8. Let $K \subseteq G(\mathbb{A}_f)$ be a compact open subgroup. The *Shimura variety of level K associated with (G, X)* is the double coset space

$$\mathrm{Sh}_K(G, X) := G(\mathbb{Q})\backslash(X \times (G(\mathbb{A}_f)/K)).$$

A priori, this is just a set, but if K is sufficiently small (i.e., “neat” in the sense of [Bor69; Pin90]), $\mathrm{Sh}_K(G, X)$ can be canonically written as a finite disjoint union of hermitian symmetric domains.[2] In particular, this gives $\mathrm{Sh}_K(G,X)$ the structure of a complex manifold.

In fact, by the theorem of Baily-Borel, this complex manifold attains the structure of a quasi-projective complex variety in a canonical way.

By work of Deligne, Milne and Borovoi, this variety is defined already (and again in a canonical way) over the reflex field $\mathbf{E}$. So in particular, it is defined over a number field independent of K. This is important when varying K and it is the reason why we consider the whole Shimura variety instead of its connected components over $\mathbb{C}$ on their own. It is possible for the Shimura variety to have multiple connected components over $\mathbb{C}$ while being connected over $\mathbf{E}$.

More detailed explanations may be found in [Mil05].

Example 1.9. We describe the Siegel Shimura variety (merely as a complex manifold for now) of level N (where $N \in \mathbb{N}$, $N > 2$), i.e., the Shimura variety associated with $(\mathrm{GSp}_{2g,\mathbb{Q}}, S^{\pm})$ (cf. Example 1.2) of level $K(N)$, where $K(N)$ is defined by the exact sequence

$$1 \to K(N) \to \mathrm{GSp}_{2g}(\hat{\mathbb{Z}}) \to \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z}) \to 1.$$

Define a moduli problem (in this case simply given as a set) $\mathsf{A}_{g,N}(\mathbb{C}) := \{(A, \lambda, \phi_N)\}/\cong$ with

- (A, λ) principally polarized abelian $\mathbb{C}$-variety,
- $\phi_N \colon ((\mathbb{Z}/N\mathbb{Z})^{2g}, \text{std. sympl. form}) \to (A(\mathbb{C})[N], \text{Weil pairing given by } \lambda)$ a symplectic similitude.

We then have

$$\begin{aligned} \mathsf{A}_{g,N}(\mathbb{C}) &\cong \coprod_{\varphi(N) \text{ copies}} \Gamma(N)\backslash\mathbb{H}_g \\ &= \mathrm{Sh}_{K(N)}(\mathrm{GSp}_{2g}, \mathbb{H}_g^{\pm})(\mathbb{C}), \end{aligned}$$

where $\Gamma(N) := \ker(\mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})) \subseteq \mathrm{Sp}_{2g}(\mathbb{Z})$ and $\mathbb{H}_g \cong S^+$ is the Siegel upper half-space of degree g.

For $g = 1$ we now describe how to get from the moduli problem to the adelic double quotient in more detail; the general case works similarly.

[2] If K fails to be sufficiently small, one might very reasonably argue that our definition of the Shimura variety of level K really is the definition of the *coarse* Shimura variety and that one should be working with stacks instead. Since we will only be interested in sufficiently small level, this is inconsequential for us.

With

$$\mathrm{Lat}_2^N(\mathbb{Z},\mathbb{C}) := \left\{ \begin{matrix} \text{full rank 2 lattices } \Lambda \text{ in } \mathbb{C} \text{ together} \\ \text{with an isomorphism } (\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{\sim} \frac{1}{N}\Lambda/\Lambda \end{matrix} \right\},$$

$$\mathrm{CS}(V) := \{\text{complex structures on the real vector space } V\},$$

and $\mathrm{Lat}_N^2(\mathbb{Z},\mathbb{R}^2), \mathrm{Lat}_2^N(\mathbb{Z},\mathbb{Q}^2), \mathrm{Lat}_2^N(\hat{\mathbb{Z}},\mathbb{A}_f^2)$ defined analogously to $\mathrm{Lat}_2^N(\mathbb{Z},\mathbb{C})$, we have a sequence of isomorphisms

$$\begin{aligned}
\mathsf{A}_{1,N}(\mathbb{C}) &\cong \cong\backslash \mathrm{Lat}_N^2(\mathbb{Z},\mathbb{C}) \cong \bigsqcup_{\varphi(N) \text{ copies}} \Gamma(N)\backslash\mathbb{H} \\
&\cong \cong\backslash(\mathrm{CS}(\mathbb{R}^2) \times \mathrm{Lat}_N^2(\mathbb{Z},\mathbb{R}^2)) \\
&\cong \mathrm{GL}_2(\mathbb{R})\backslash(\mathrm{CS}(\mathbb{R}^2) \times \mathrm{Lat}_2^N(\mathbb{Z},\mathbb{R}^2)) \\
&\cong \mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{CS}(\mathbb{R}^2) \times \mathrm{Lat}_2^N(\mathbb{Z},\mathbb{Q}^2)) \\
&\cong \mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{CS}(\mathbb{R}^2) \times \mathrm{Lat}_2^N(\hat{\mathbb{Z}},\mathbb{A}_f^2)) \\
&\cong \mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{CS}(\mathbb{R}^2) \times (\mathrm{GL}_2(\mathbb{A}_f)/K(N))) \\
&\cong \mathrm{GL}_2(\mathbb{Q})\backslash(\mathbb{H}^\pm \times (\mathrm{GL}_2(\mathbb{A}_f)/K(N))) \\
&= \mathrm{Sh}_{K(N)}(\mathrm{GL}_2,\mathbb{H}^\pm)(\mathbb{C}).
\end{aligned}$$

1.2 Bruhat-Tits buildings

Let K be a complete discrete valuation field with ring of integers $\mathcal{O}$, uniformizer ϖ and perfect residue field $\kappa := \mathcal{O}/\varpi$.

Notation 1.10. For a (connected) reductive group G over K, we denote by $\mathcal{B}(G,K)$ the extended (or enlarged) and by $\mathcal{B}^{\mathrm{red}}(G,K)$ the reduced (i.e., non-extended) Bruhat-Tits building of G over K [BT84a]. Moreover, $\mathcal{B}^{\mathrm{abstract}}(G,K)$ denotes the underlying abstract simplicial complex.

Remark 1.11. Let V be a finite-dimensional K-vector space.

As described in [KP15, 1.1.9] (originally in [BT84b]), the points of $\mathcal{B}(\mathrm{GL}(V),K)$ correspond to graded periodic lattice chains $(\mathcal{L},c)$, i.e.,

- $\emptyset \neq \mathcal{L}$ is a totally ordered set of full $\mathcal{O}$-lattices in V stable under scalar multiplication (i.e., $\Lambda \in \mathcal{L} \iff \varpi\Lambda \in \mathcal{L}$),

- $c\colon \mathcal{L} \to \mathbb{R}$ is a strictly decreasing function such that $c(\varpi^n \Lambda) = c(\Lambda) + n$.

Remark 1.12. Fix such a $\mathcal{L}$ and let $\Lambda^0 \in \mathcal{L}$. Then every homothety class of lattices has a unique representative Λ such that $\Lambda \subseteq \Lambda^0$ and $\Lambda \not\subseteq \varpi\Lambda^0$. Consider such representatives Λ^i for all of the distinct homothety classes of lattices that make up $\mathcal{L}$. Because $\mathcal{L}$ is totally ordered and $\Lambda^i \not\subseteq \varpi\Lambda^0$, it follows that $\Lambda^i \supseteq \varpi\Lambda^0$ for all i and that $\{\Lambda^i/\varpi\Lambda^0\}_i$ is a flag of non-trivial linear subspaces of $\Lambda^0/\varpi\Lambda^0 \cong \kappa^n$, where $n := \dim V$. Consequently, the number r of homothety classes is in $\{1, \dots, n\}$; it is called the *period length* (or *rank*) of $\mathcal{L}$. Numbering the Λ^i in descending order we hence obtain r lattices $\Lambda^0, \Lambda^1, \dots, \Lambda^{r-1}$ such that

$$\Lambda^0 \supsetneqq \Lambda^1 \supsetneqq \cdots \supsetneqq \Lambda^{r-1} \supsetneqq \varpi\Lambda^0 \tag{1.13}$$

and $\mathcal{L}$ is given by the the strictly descending sequence of lattices

$$\Lambda^{qr+i} = \varpi^q \Lambda^i, \quad q \in \mathbb{Z},\ 0 \le i < r.$$

Remark 1.14. Let V be a finite-dimensional symplectic K-vector space.

$\mathcal{B}(\mathrm{GSp}(V), K)$ embeds into the subset of $\mathcal{B}(\mathrm{GL}(V), K)$ consisting of those $(\mathcal{L}, c)$ such that $\Lambda \in \mathcal{L} \implies \Lambda^\vee \in \mathcal{L}$.

Passing to the underlying abstract simplicial complexes means forgetting about the grading c and

$$\mathcal{B}^{\text{abstract}}(\mathrm{GSp}(V), K) = \{\mathcal{L} \in \mathcal{B}^{\text{abstract}}(\mathrm{GL}(V), K) \mid \Lambda \in \mathcal{L} \implies \Lambda^\vee \in \mathcal{L}\}.$$

If $\mathcal{L} \in \mathcal{B}^{\text{abstract}}(\mathrm{GSp}(V), K)$ and $\{\Lambda^i\}_i$ is as in Remark 1.12, then there is an involution $t\colon \mathbb{Z} \to \mathbb{Z}$ with $(\Lambda^i)^\vee = \Lambda^{t(i)}$, $t(i + qr) = t(i) - qr$, and $i < j \implies t(i) > t(j)$. So $-a := t(0) > t(1) > \cdots > t(r) = -a - r$, which implies $t(i) = -i - a$. Thus $i_0 - t(i_0) = 2i_0 + a \in \{0, 1\}$ for some unique $i_0 \in \mathbb{Z}$. Hence, upon renumbering the Λ^i, we may assume that $a \in \{0, 1\}$.

We therefore have

$$\begin{aligned} \varpi\Lambda^0 \subsetneqq \Lambda^{r-1} \subsetneqq \Lambda^{r-2} \subsetneqq \cdots \subsetneqq \Lambda^0 \subseteq \left(\Lambda^0\right)^\vee = \Lambda^{-a} \subsetneqq \left(\Lambda^1\right)^\vee = \Lambda^{-1-a} \\ \subsetneqq \cdots \subsetneqq \left(\Lambda^{r-1}\right)^\vee = \Lambda^{-r+1-a} \subseteq \Lambda^{-r} = \varpi^{-1}\Lambda^0. \end{aligned}$$

Example 1.15. See also section 3.2.5 for some elaborations on the building of $\mathrm{GSp}_4(\mathbb{Q}_p)$.

1.3 Alteration of the Hodge embedding

Notation 1.16. Let E be a finite field extension of $\mathbb{Q}_p$.

Denote by $\breve{E}$ the completion of the maximal unramified extension of E (hence $\breve{E} = E \cdot \breve{\mathbb{Q}}_p$).

Remark 1.17. If $E/\mathbb{Q}_p$ is unramified, then $\mathcal{O}_{\breve{E}} = W(\bar{\mathbb{F}}_p)$, $\bar{\mathbb{F}}_p$ denoting an algebraic closure of $\mathbb{F}_p$ and $W \colon \mathrm{Ring} \to \mathrm{Ring}$ being the (p-adic) Witt vectors functor. This generalizes to the ramified case using *ramified Witt vectors* instead, see e.g. [Haz78, Chap. IV, (18.6.13)] or [Ahs11, Chapter 1].

Let (G, X) be a Shimura datum of Hodge type, let $(G, X) \hookrightarrow (\mathrm{GSp}(V), S^{\pm})$ be an embedding as in Definition 1.1 (4), and let $x \in \mathcal{B}(G, \mathbb{Q}_p)$ be a point in the Bruhat-Tits building of G over $\mathbb{Q}_p$.

We consider the associated Bruhat-Tits scheme $\mathcal{G}_x$, i.e., the affine smooth model of $G_{\mathbb{Q}_p}$ over $\mathbb{Z}_p$ such that $\mathcal{G}_x(\breve{\mathbb{Z}}_p) \subseteq G(\breve{\mathbb{Q}}_p)$ is the stabilizer of the facet of x in $\mathcal{B}(G, \breve{\mathbb{Q}}_p) \overset{\text{[Lan00, Prop. 2.1.3]}}{=} \mathcal{B}(G, \mathbb{Q}_p^{\mathrm{ur}})$. Let $K_p := \mathcal{G}_x(\mathbb{Z}_p) \subseteq G(\mathbb{Q}_p)$ and let $K^p \subseteq G(\mathbb{A}_f^p)$ be a sufficiently small open compact subgroup. Define $K := K_p K^p \subseteq G(\mathbb{A}_f)$.

Assumptions 1.18. From now on, we will always make the following assumptions:

- $\mathcal{G}_x = \mathcal{G}_x^\circ$ is connected.
- G splits over a tamely ramified extension of $\mathbb{Q}_p$.
- $p \nmid \#\pi_1(G^{\mathrm{der}})$.

Notation 1.19. In order not to make notation overly cumbersome, we usually denote the base change $G_{\mathbb{Q}_p}$ of G to $\mathbb{Q}_p$ by G again. (Later, we will almost exclusively be dealing with $G_{\mathbb{Q}_p}$.)

Under the above assumptions, Kisin and Pappas construct in [KP15, section 1.2] (building on [Lan00]) a toral $G(\breve{\mathbb{Q}}_p)$- and $\mathrm{Gal}(\breve{\mathbb{Q}}_p/\mathbb{Q}_p)$-equivariant embedding

$$\iota \colon \mathcal{B}(G, \breve{\mathbb{Q}}_p) \to \mathcal{B}(\mathrm{GL}(V), \breve{\mathbb{Q}}_p),$$

restricting[3] to a map $\iota \colon \mathcal{B}(G, \mathbb{Q}_p) \to \mathcal{B}(\mathrm{GL}(V), \mathbb{Q}_p)$.

[3] In this case, this is obvious from Galois-equivariance, since $\mathcal{B}(G, K) = \mathcal{B}(G, K')^{\mathrm{Gal}(K'/K)}$ if K'/K is an unramified extension.

ι being a toral embedding means the following: ι is isometric after a suitable normalization of the norm on $\mathcal{B}(G, \breve{\mathbb{Q}}_p)$; moreover it is the canonical extension of a map $\mathcal{B}^{\mathrm{red}}(G, \breve{\mathbb{Q}}_p) \to \mathcal{B}^{\mathrm{red}}(\mathrm{GL}(V), \breve{\mathbb{Q}}_p)$ such that for each maximal $\breve{\mathbb{Q}}_p$-split torus $S \subseteq G$ there exists a maximal $\breve{\mathbb{Q}}_p$-split torus $T \subseteq \mathrm{GL}(V)$ such that $G \hookrightarrow \mathrm{GL}(V)$ maps S into T and ι restricts to a map between the reduced apartments associated with (G, S) and $(\mathrm{GL}(V), T)$, respectively, compatible with translations.

This embedding ι depends on some choices: By assumption, there exists a tamely ramified Galois extension $\tilde{K}/\mathbb{Q}_p$ with finite inertia group such that $G_{\tilde{K}}$ is split reductive. Let $H \to \operatorname{Spec} \mathbb{Z}_p$ be the split Chevalley form of G over $\mathbb{Z}_p$. Then ι depends on the choice of

- an isomorphism $G_{\tilde{K}} \cong H_{\tilde{K}}$,
- a pinning $(T, M, f, R, \Delta, (X_\alpha)_{\alpha \in \Delta})$ of H (cf. [SGA3, Exp. XXIII][4]), which also entails the following: Let B be the Borel subgroup of H corresponding to Δ. When we talk about roots, it is with respect to T, and when we talk about positive roots and so on, it is with respect to (T, B). We also fix a hyperspecial vertex x_o in $\mathcal{B}(H, \mathbb{Q}_p)$ with stabilizer $H(\mathbb{Z}_p)$.
- for every $\mathbb{Q}_p$-irreducible summand[5] V_i of the representation $G_{\mathbb{Q}_p} \to \mathrm{GL}(V)_{\mathbb{Q}_p}$ a lattice $\Lambda_i = U(\mathfrak{n}^-)v_i \subseteq V_i$, where $v_i \neq 0$ is a highest weight vector of V_i and $\mathfrak{n}^-$ is the (strictly) negative root space inside the Lie algebra of H over $\mathbb{Z}_p$,
- a grading $c_{\Lambda_i} + t_i$ of the lattice chain $\{p^n \Lambda_i\}_{n \in \mathbb{Z}}$ given by real numbers $t_i \in \mathbb{R}$. Here $(c_{\Lambda_i} + t_i)(p^n \Lambda_i) := n + t_i$.

By [KP15, Lemma 2.3.3], we can (and do) arrange these choices in such a way that ι factors

$$\mathcal{B}(G, \breve{\mathbb{Q}}_p) \xrightarrow{j} \mathcal{B}(\mathrm{GSp}(V), \breve{\mathbb{Q}}_p) \to \mathcal{B}(\mathrm{GL}(V), \breve{\mathbb{Q}}_p),$$

where the last map is the canonical toral embedding (whose definition will be clear from Remark 1.14). Again, j restricts to a map

$$j \colon \mathcal{B}(G, \mathbb{Q}_p) \hookrightarrow \mathcal{B}(\mathrm{GSp}(V), \mathbb{Q}_p). \tag{1.20}$$

Let $y = (\mathcal{L}, c)$ be the image of $x \in \mathcal{B}(G, \mathbb{Q}_p)$ under the map (1.20) and let $(\Lambda^i)_i, r, a$ be as in Remark 1.14. We define $N_p := \mathrm{Stab}_{\mathrm{GSp}(V)(\mathbb{Q}_p)}(\mathcal{L})$.

[4]Since $\operatorname{Spec} \mathbb{Z}_p$ is connected, some technicalities from *loc. cit.* disappear here.

[5]Note that by reductivity and $\mathrm{char}(\mathbb{Q}_p) = 0$, every finite-dimensional representation of $G_{\mathbb{Q}_p}$ is completely reducible.

Consider the symplectic $\mathbb{Q}$-vector space

$$V^{\S} := \bigoplus_{i=-(r-1)-a}^{r-1} V$$

(direct sum of symplectic spaces, i.e., if ψ denotes the symplectic form on V, then the symplectic form $\psi^{\S}$ on $V^{\S}$ is given by $\bigoplus_{i=-(r-1)-a}^{r-1} \psi$) and the lattice in $V^{\S}_{\mathbb{Q}_p}$

$$\Lambda^{\S} := \bigoplus_{i=-(r-1)-a}^{r-1} \Lambda^i.$$

By replacing $\Lambda^{\S}$ by a homothetic lattice, we may assume that $\Lambda^{\S} \subseteq \left(\Lambda^{\S}\right)^{\vee}$ (hence $\left(\Lambda^{\S}\right)^{\vee\vee} = \Lambda^{\S}$).

We have a diagonal embedding

$$\mathrm{GSp}(V) \hookrightarrow \mathrm{GSp}(V^{\S})$$

and (with calligraphic script meaning that we talk about Bruhat-Tits group schemes)

$$\mathcal{GSP}(V)_y = \mathrm{Stab}_{\mathrm{GSp}(V)}(\mathcal{L}) = \bigcap \mathrm{Stab}_{\mathrm{GSp}(V)}(\Lambda^i) \subseteq \mathrm{GSp}(\Lambda^{\S}) \subseteq \mathrm{GL}(\Lambda^{\S}) \subseteq \mathcal{GL}(V)_{\Lambda^{\S}},$$

so that upon replacing our original embedding $G \hookrightarrow \mathrm{GSp}(V)$ by the embedding $G \hookrightarrow \mathrm{GSp}(V) \hookrightarrow \mathrm{GSp}(V^{\S})$, we can assume that the parahoric subgroup on the Siegel side is given as

$$\mathrm{Stab}_{\mathrm{GL}(V^{\S})(\mathbb{Q}_p)}(\Lambda^{\S}) \cap \mathrm{GSp}(V^{\S})(\mathbb{Q}_p),$$

i.e., that our level is essentially given by a single lattice, albeit one that is (in general) not self-dual. This is [KP15, 2.3.15].

1.4 Siegel integral models

With notation as above let

$$\begin{aligned} N_p &:= \mathrm{Stab}_{\mathrm{GSp}(V)(\mathbb{Q}_p)}(\mathcal{L}) \quad \text{(as before)}, \\ J_p &:= \mathrm{Stab}_{\mathrm{GL}(V^{\S})(\mathbb{Q}_p)}(\Lambda^{\S}) \cap \mathrm{GSp}(V^{\S})(\mathbb{Q}_p). \end{aligned}$$

Let $N^p \subseteq \mathrm{GSp}(V)(\mathbb{A}_f^p)$ and $J^p \subseteq \mathrm{GSp}(V^{\S})(\mathbb{A}_f^p)$ be sufficiently small open compact subgroups, and $N := N_p N^p$, $J := J_p J^p$.

In this subsection, we are going to describe integral models of $\mathrm{Sh}_N(\mathrm{GSp}(V), S^{\pm})$ and of $\mathrm{Sh}_J(\mathrm{GSp}(V^{\S}), S^{\S,\pm})$ over $\mathbb{Z}_{(p)}$ and relate the two.

Remark 1.21. By [RZ96, Definition 6.9], the integral model $\mathscr{S}_N(\mathrm{GSp}(V), S^\pm)$ is given by the moduli problem $(\mathbb{Z}_{(p)}\text{-scheme}) \ni S \mapsto \left\{(A, \bar{\lambda}, \eta^p)\right\}/\cong$, where:

(a) $A = (A_\Lambda)_{\Lambda \in \mathcal{L}}$ is an $\mathcal{L}$-set of abelian schemes, i.e.,

- for every $\Lambda \in \mathcal{L}$, an abelian S-scheme up to $\mathbb{Z}_{(p)}$-isogeny A_Λ (i.e., A_Λ is an object of the category $(\text{abelian } S\text{-schemes}) \otimes \mathbb{Z}_{(p)}$, where the category $\mathcal{A} \otimes R$ for $\mathcal{A}$ an preadditive category and R a ring has the same objects as $\mathcal{A}$ and $\mathrm{Hom}_{\mathcal{A}\otimes R}(X, Y) = \mathrm{Hom}(X, Y) \otimes_{\mathbb{Z}} R$ for all objects X, Y),
- for every inclusion $\Lambda_1 \subseteq \Lambda_2$ a $\mathbb{Z}_{(p)}$-isogeny $\rho_{\Lambda_2,\Lambda_1} \colon A_{\Lambda_1} \to A_{\Lambda_2}$,
- $\rho_{\Lambda_3,\Lambda_1} = \rho_{\Lambda_3,\Lambda_2} \circ \rho_{\Lambda_2,\Lambda_1}$ if $\Lambda_1 \subseteq \Lambda_2 \subseteq \Lambda_3$ in $\mathcal{L}$,
- the height of $\rho_{\Lambda_2,\Lambda_1}$ is $\log_p |\Lambda_2/\Lambda_1|$. Here $\rho_{\Lambda_2,\Lambda_1}$ gives rise to a well-defined homomorphism of p-divisible groups, and what we mean is that the kernel of this homomorphism (which is a finite locally free commutative group scheme, which we also refer to simply as the kernel of $\rho_{\Lambda_2,\Lambda_1}$) is to have order $|\Lambda_2/\Lambda_1|$.
- For every $\Lambda \in \mathcal{L}$, there is an isomorphism (called *periodicity isomorphism*) $\theta_\Lambda \colon A_\Lambda \to A_{p\Lambda}$ such that $\rho_{\Lambda,p\Lambda} \circ \theta_\Lambda = [p] \colon A_\Lambda \to A_\Lambda$ is the multiplication-by-p isogeny.

(b) $\bar{\lambda} \colon A \to \tilde{A}$ is a $\mathbb{Q}$-homogeneous principal polarization, i.e., a $\underline{\mathbb{Q}^\times}$-orbit of a principal polarization $\lambda \colon A \to \tilde{A}$. Here $\tilde{A}$ is the $\mathcal{L}$-set of abelian schemes over S up to prime-to-p isogeny given by $\tilde{A}_\Lambda := (A_{\Lambda^\vee})^\vee$. And being a polarization λ means being a quasi-isogeny of $\mathcal{L}$-sets $\lambda \colon A \to \tilde{A}$ such that

$$A_\Lambda \xrightarrow{\lambda_\Lambda} \tilde{A}_\Lambda = (A_{\Lambda^\vee})^\vee \xrightarrow{\varrho^\vee_{\Lambda^\vee,\Lambda}} (A_\Lambda)^\vee$$

is a polarization of A_Λ for all Λ. If λ_Λ can be chosen to be an isomorphism up to prime-to-p isogeny for all Λ, then we speak of a principal polarization. In that case, when referring to λ_Λ, we mean a λ_Λ which is an isomorphism up to prime-to-p isogeny.

(c) η^p is a level-N^p-structure, i.e. (if S is connected), it is a $\pi_1(S, s)$-invariant N^p-orbit of symplectic similitudes $\eta^p \colon V_{\mathbb{A}_f^p} \to H_1(A_s, \mathbb{A}_f^p)$ (where s is some geometric basepoint and $H_1(A_s, \mathbb{A}_f^p)$ with its $\pi_1(S, s)$-action corresponds to the Tate $\mathbb{A}_f^p$-module of A (cf. [RZ96, 6.8]), which is a smooth $\mathbb{A}_f^p$-sheaf). Note that this forces the abelian schemes A_Λ to be $(\dim_{\mathbb{Q}} V)$-dimensional.

Definition 1.22. Set $\Lambda^{\S}_{\mathbb{Z}_{(p)}} := \Lambda^{\S}_{\mathbb{Z}_p} \cap V^{\S}_{\mathbb{Q}} = \prod_{i=-(r-1)-a}^{r-1} \Lambda^{i}_{\mathbb{Z}_{(p)}}$. We choose a lattice $\Lambda^{\S}_{\mathbb{Z}} \subseteq V^{\S}$ such that $\Lambda^{\S}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = \Lambda^{\S}_{\mathbb{Z}_{(p)}}$ and $\Lambda^{\S}_{\mathbb{Z}} \subseteq (\Lambda^{\S}_{\mathbb{Z}})^{\vee}$.

Remark 1.23. Set $d := \left|\left(\Lambda^{\S}_{\mathbb{Z}}\right)^{\vee} / \Lambda^{\S}_{\mathbb{Z}}\right|$. By [Kis10, 2.3.3, 3.2.4], the integral model $\mathscr{S}_J(\mathrm{GSp}(V^{\S}), S^{\S,\pm})$ is given by the moduli problem $(\mathbb{Z}_{(p)}\text{-schemes}) \ni S \mapsto \left\{(A^{\S}, \lambda^{\S}, \epsilon^p)\right\}/\cong$, where

(a) $A^{\S}$ is an abelian scheme over S up to $\mathbb{Z}_{(p)}$-isogeny,

(b) $\lambda^{\S}\colon A^{\S} \to \left(A^{\S}\right)^{\vee}$ is a polarization of degree d (i.e., the polarization of the (well-defined) associated p-divisible group has degree d),

(c) ϵ^p is a level-J^p-structure, i.e. (if S is connected), it is a $\pi_1(S, s)$-invariant J^p-orbit of symplectic similitudes $\epsilon^p\colon V^{\S}_{\mathbb{A}_f^p} \to H_1(A^{\S}_s, \mathbb{A}_f^p)$. Note that this forces the abelian schemes $A^{\S}$ to be $(\dim_{\mathbb{Q}} V^{\S})$-dimensional.

This completes the descriptions of the moduli problems, and we turn to the question of the relationship between the two. Consider (for appropriate N^p, J^p; see below) the morphism $\chi\colon \mathscr{S}_N(\mathrm{GSp}(V), S^{\pm}) \to \mathscr{S}_J(\mathrm{GSp}(V^{\S}), S^{\S,\pm})$ given on S-valued points by sending $(A, \bar{\lambda}, \eta^p)$ to $(A^{\S}, \lambda^{\S}, \epsilon^p)$, where

(a) $A^{\S} := \displaystyle\prod_{i=-(r-1)-a}^{r-1} A_{\Lambda^i}$,

(b) $\lambda^{\S} := \displaystyle\prod_{i=-(r-1)-a}^{r-1} \left(\rho^{\vee}_{(\Lambda^i)^{\vee}, \Lambda^i} \circ \lambda_{\Lambda^i}\right)$,

(c) ϵ^p is the product $\prod_{i=-(r-1)-a}^{r-1} \eta^p$, to be interpreted as the product over $\eta^p\colon V_{\mathbb{A}_f^p} \to H_1(A_{\Lambda^i,s}, \mathbb{A}_f^p) \cong H_1(A_s, \mathbb{A}_f^p)$, where the isomorphism $H_1(A_{\Lambda^i,s}, \mathbb{A}_f^p) \cong H_1(A_s, \mathbb{A}_f^p)$ is by definition the identity for some fixed $i = i_0$ and otherwise induced by the transition map $\rho_{\Lambda^i, \Lambda^{i_0}}$. We need that N^p is mapped into J^p by $\mathrm{GSp}(V) \hookrightarrow \mathrm{GSp}(V^{\S})$ for this to make sense.

Lemma 1.24. *Let S be a scheme, $\ell \neq p$ prime numbers. If ℓ does not appear as a residue characteristic of S, then the Tate module functors*

$$H_1(\_, \mathbb{Z}_\ell)\colon (\textit{abelian } S\textit{-schemes}) \to (\textit{étale } \mathbb{Z}_\ell\textit{-local systems on } S),$$
$$H_1(\_, \mathbb{Q}_\ell)\colon (\textit{abelian } S\textit{-schemes}) \to (\textit{étale } \mathbb{Q}_\ell\textit{-local systems on } S)$$

(cf. [Gro74, III, 5.4 and 6.2] for precise definitions) are faithful.

If only p and 0 appear as residue characteristics of S, then the Tate module functor

$$H_1(\_, \mathbb{A}_f^p) \colon (\text{abelian } S\text{-schemes}) \to (\text{étale } \mathbb{A}_f^p\text{-local systems on } S)$$

is faithful.

Proof: First note that the statements about $H_1(\_, \mathbb{Q}_\ell)$ and $H_1(\_, \mathbb{A}_f^p)$ follows from the statement about $H_1(\_, \mathbb{Z}_\ell)$, which is why it is enough to only look at $H_1(\_, \mathbb{Z}_\ell)$.

A homomorphism of abelian S-schemes $f \colon A \to B$ vanishes if and only if it vanishes over every (geometric) fiber of S: Indeed, if it vanishes fiberwise, then it is flat by the fiber criterion for flatness. Applying that criterion again we see that the closed immersion and fiberwise isomorphism $\ker(f) \hookrightarrow A$ is flat, which means that is an isomorphism.

This way we are reduced to the case where R is an (algebraically closed) field of characteristic different from ℓ. In this setting the faithfulness is well-known (the salient point being that the ℓ-primary torsion is dense).

Lemma 1.25. *Let H be a totally disconnected locally compact*[6] *group (i.e., a locally profinite group) and let $N \subseteq H$ be a compact subgroup. Then*

$$N = \bigcap_{\substack{N \subseteq J \\ J \subseteq H \text{ open compact subgrp.}}} J.$$

Note that this is (a variant of) a well-known theorem by van Dantzig if $N = \{1\}$ [Dan36].

Proof: We make use of the following fact [AT08, Prop. 3.1.7]: A Hausdorff space is locally compact and totally disconnected if and only if the open compact sets form a basis of the topology. (Van Dantzig's theorem is the group version of this, which talks only about a neighborhood basis of the identity and open compact *subgroups*.)

First we show that N is contained in some open compact subset $K \subseteq H$. For every $x \in N$ choose a compact open neighborhood $x \in K_x \subseteq H$. This is possible by the fact cited above. Then there is a finite subset $I \subseteq N$ such that $N \subseteq \bigcup_{x \in I} K_x =: K$.

Next, for every $x \in N$ choose an open neighborhood of the identity U_x such that $xU_xK \subseteq K$. With $N \subseteq U := \bigcup_{x \in N} xU_x$ we obtain $UK \subseteq K$. Replacing U by $U \cap U^{-1}$,

[6]By (our) definition, locally compact implies Hausdorff.

we may moreover assume it is symmetric. The subgroup generated by U is open (hence closed) and contained in K, hence is an open compact subgroup.

Thus N even is contained in an open compact sub*group*; in other words, we may assume that H is compact, i.e., is a profinite group.

Then H/N is compact[7] and totally disconnected[8] (i.e., is a Stone space). By the fact cited above,

$$H/N \supseteq \{1\} = \bigcap_{L \subseteq H/N \text{ open compact subset}} L.$$

Observe that the quotient map $H \to H/N$ is proper to deduce

$$N = \bigcap_{\substack{N \subseteq M \\ M \subseteq H \text{ open compact subset}}} M.$$

Say M is an open and compact subset of H containing N. As we have shown above, there is an open compact subgroup $J \subseteq H$ in between N and M, and this is all we need to complete the proof. □

Proposition 1.26. *For every compact open subgroup $N^p \subseteq \mathrm{GSp}(V)(\mathbb{A}_f^p)$*

$$\chi\colon \mathscr{S}_N(\mathrm{GSp}(V), S^{\pm}) \to \mathscr{S}_J(\mathrm{GSp}(V^{\S}), S^{\S,\pm})$$

is a well-defined morphism for all compact open subgroups $N^p \subseteq J^p \subseteq \mathrm{GSp}(V^{\S})(\mathbb{A}_f^p)$ and is a closed immersion for all sufficiently small compact open subgroups $N^p \subseteq J^p \subseteq \mathrm{GSp}(V^{\S})(\mathbb{A}_f^p)$.

Proof: The fact that it's well-defined is clear from the construction.

To show the second statement, as in [Del71, Prop. 1.15], it is enough to show that

$$\mathscr{S}_{N_pN^p}(\mathrm{GSp}(V), S^{\pm}) \to \varprojlim_{J^p} \mathscr{S}_{J_pJ^p}(\mathrm{GSp}(V^{\S}), S^{\S,\pm})$$

is a closed immersion, i.e., a proper monomorphism.

[7] Hausdorff quotient spaces of compact spaces are compact again, but for "locally compact" the analogous statement is not true in general!

[8] Take $x, y \in H$ such that $xN \neq yN$. We show that any subspace $S \subseteq H/N$ containing both xN and yN is disconnected. Let $U \subseteq H/N$ be a neighborhood of xN not containing yN. Let $x \in V \subseteq \pi^{-1}(U)$ be open and compact, where $\pi\colon H \to H/N$ is the projection. Then $yN \notin \pi(V) \subseteq H/N$ is open and compact (hence closed) and we have $S = (\pi(V) \cap S) \sqcup S \setminus \pi(V)$ where both $\pi(V) \cap S$ and $S \setminus \pi(V)$ are open in S. This shows that S is disconnected.

We begin by proving that it is a monomorphism, i.e., injective on S-valued points (S arbitrary $\mathbb{Z}_{(p)}$-scheme). So, say $(A_1, \lambda_1, \eta_1^p)$ and $(A_2, \lambda_2, \eta_2^p)$ both map to $(A^\S, \lambda^\S, \epsilon_{J^p}^p)$. That means precisely that there is an isomorphism of abelian S-schemes up to $\mathbb{Z}_{(p)}$-isogeny

$$\phi\colon \prod_{i=-(r-1)-a}^{r-1} A_{1,\Lambda^i} \xrightarrow{\cong} \prod_{i=-(r-1)-a}^{r-1} A_{2,\Lambda^i}$$

such that

$$\phi^\vee \circ \prod_{i=-(r-1)-a}^{r-1} \left(\rho_{2,(\Lambda^i)^\vee,\Lambda^i}^\vee \circ \lambda_{2,\Lambda^i}\right) \circ \phi = \prod_{i=-(r-1)-a}^{r-1} \left(\rho_{1,(\Lambda^i)^\vee,\Lambda^i}^\vee \circ \lambda_{1,\Lambda^i}\right)$$

and

$$H_1(\phi, \mathbb{A}_f^p) \circ \epsilon_{1,J^p}^p = \epsilon_{2,J^p}^p \mod J^p.$$

We claim that ϕ comes from isomorphisms

$$\phi_i\colon A_{1,\Lambda^i} \xrightarrow{\cong} A_{2,\Lambda^i}.$$

Certainly there is but one candidate for ϕ_i: define ϕ_i to be the composition

$$A_{1,\Lambda^i} \xrightarrow{\text{incl}} \prod_{i=-(r-1)-a}^{r-1} A_{1,\Lambda^i} \xrightarrow{\phi} \prod_{i=-(r-1)-a}^{r-1} A_{2,\Lambda^i} \xrightarrow{\text{proj}} A_{2,\Lambda^i}.$$

Our claim then is that

$$\phi = \prod_{i=-(r-1)-a}^{r-1} \phi_i.$$

Apply $H^1(\_, \mathbb{A}_f^p)$ on both sides. For the left hand side, we have

$$H_1(\phi, \mathbb{A}_f^p) = \epsilon_{2,J^p}^p \circ \left(\epsilon_{1,J^p}^p\right)^{-1} \mod J^p.$$

and the right hand side of this equation is block diagonal. So

$$H_1(\phi, \mathbb{A}_f^p) = \prod_{i=-(r-1)-a}^{r-1} H_1(\phi_i, \mathbb{A}_f^p) \mod J^p.$$

Since (by Lemma 1.25)

$$N^p = \bigcap_{\substack{N_\ell \subseteq J_\ell \\ J_\ell \subseteq \mathrm{GSp}(V^\S)(\mathbb{Q}_\ell) \text{ cpt. open subgrp.}}} J_\ell,$$

it follows that (with $\ell \neq p$)

$$H_1(\phi, \mathbb{Q}_\ell) = \prod_{i=-(r-1)-a}^{r-1} H_1(\phi_i, \mathbb{Q}_\ell) \mod N_\ell,$$

hence (since N_ℓ acts block-diagonally) that $H_1(\phi, \mathbb{Q}_\ell) = \prod_{i=-(r-1)-a}^{r-1} H_1(\phi_i, \mathbb{Q}_\ell)$.

Since $H_1(\_, \mathbb{Q}_\ell)$ is faithful (Lemma 1.24), this implies $\phi = \prod_{i=-(r-1)-a}^{r-1} \phi_i$, as desired.

Next, consider the extension by zero of $\left(H_1(\rho_{1/2,\Lambda^j,\Lambda^i}, \mathbb{A}_f^p)\right)_{i,j}$ (where for "$1/2$" either "1" or "2" can be plugged in) to a map $H_1(A^\S, \mathbb{A}_f^p) \to H_1(A^\S, \mathbb{A}_f^p)$. Under the isomorphism given by the J^p-level structure this corresponds, up to the J^p-action, to the map $V^\S_{\mathbb{A}_f^p} \to V^\S_{\mathbb{A}_f^p}$ given by mapping the i'th copy of $V_{\mathbb{A}_f^p}$ identically to the j'th copy and the rest to zero. Thus $\rho_{1/2,i,j}$ yield the same up to J^p after applying $H_1(\_, \mathbb{A}_f^p)$, hence they are equal in the $\mathbb{Z}_{(p)}$-isogeny category.

Consequently, χ is a monomorphism.

For properness, we will use the valuative criterion. Let R be a discrete valuation ring with field of fractions K and assume that a K-point $A^\S = \prod_{i=-(r-1)-a}^{r-1} A_{\Lambda^i}$ with its additional structures coming from $(A_{\Lambda^i})_i$ extends to an R-point $\mathcal{A}^\S$. Consider the map $A^\S \to A_{\Lambda^{i_0}} \to A^\S$ where the first map is a projection and the second an inclusion. By the Néron mapping property, this extends to a map $\mathcal{A}^\S \to \mathcal{A}^\S$. Define $\mathcal{A}_{\Lambda^{i_0}}$ to be the image of this map.

The Néron mapping property also allows us to extend the transition isogenies $\rho_{\Lambda^{i_0},\Lambda^{j_0}} : A_{\Lambda^{j_0}} \to A_{\Lambda^{i_0}}$, $i_0 \leq j_0$, the periodicity isomorphisms, and the polarization.

Since $\pi_1(\operatorname{Spec} K)$ surjects onto $\pi_1(\operatorname{Spec} R)$ (see [Stacks, Tag 0BQM]), extending the level structure away from p is trivial. □

1.5 Construction of the integral model

Let $\mathbf{E}$ be the reflex field of (G, X) and $v \mid p$ a place of $\mathbf{E}$.

As the first step towards the construction of the integral model, we define $\mathscr{S}_K^-(G, X) \to \operatorname{Spec} \mathcal{O}_{\mathbf{E},(v)}$ to be the closure of $\mathrm{Sh}_K(G, X)$ in $\mathscr{S}_N(\mathrm{GSp}(V), S^\pm)_{\mathcal{O}_{\mathbf{E},(v)}}$ (or, equivalently by

Proposition 1.26, in $\mathscr{S}_J(\mathrm{GSp}(V^\S), S^{\S,\pm})_{\mathcal{O}_{\mathbf{E},(v)}})$. By this we mean the topological closure with the reduced subscheme structure or, equivalently (since $\mathrm{Sh}_K(G,X)$ is reduced), the flat closure (in the sense of [EGA4, § 2.8]).

Then we define $\mathscr{S}_K(G,X) \to \operatorname{Spec} \mathcal{O}_{\mathbf{E},(v)}$ to be the normalization of $\mathscr{S}_K^-(G,X) \to \operatorname{Spec} \mathcal{O}_{\mathbf{E},(v)}$.

This can be regarded as "the obvious definition", but it is entirely non-obvious and due to Kisin and Pappas that this really behaves as one would expect from a canonical integral model. These expectations, properly explained in [KP15], mainly concern the satisfaction of an extension property which is a weaker version of the valuative criterion for properness, and the existence of a local model diagram that essentially shows that the integral model is étale-locally isomorphic to the local model described by Pappas and Zhu [PZ13]. We will discuss the local model in Section 1.9.

Define $E := \mathbf{E}_v$, the v-adic completion of $\mathbf{E}$, and denote by κ the residue field of E, a finite extension of $\mathbb{F}_p$. By abuse of notation, we also denote the base change of $\mathscr{S}_K(G,X)$ to $\mathcal{O}_E$ by $\mathscr{S}_K(G,X)$ again. By $\overline{\mathscr{S}}_K(G,X)$ we denote the base change of $\mathscr{S}_K(G,X)$ to κ.

1.6 Maps between Shimura varieties

This section is based on [Zho18], where more detailed explanations may be found. We let $K_p' = \mathcal{G}'(\mathbb{Z}_p)$ be another parahoric where $\mathcal{G}'$ is a Bruhat-Tits group scheme with $\mathcal{G}' = (\mathcal{G}')^\circ$. We assume that $K_p \subseteq K_p'$, i.e., that the facet (in the Bruhat-Tits building) associated with K_p' is contained in the closure of the one associated with K_p.

Theorem 1.27. [Zho18, Theorem 7.1] *For sufficiently small K^p there exists a morphism*

$$\pi_{K_pK^p,K_p'K^p} \colon \mathscr{S}_{K_pK^p}(G,X) \to \mathscr{S}_{K_p'K^p}(G,X).$$

In moving towards a proof, let us first note that $\mathrm{GSp}(V,\psi) \to \mathrm{GSp}(V^\S,\psi^\S)$ factors through

$$M := \left\{ (g_i)_i \in \prod_{i=-(r-1)-a}^{r-1} \mathrm{GSp}(V,\psi) \;\middle|\; c(g_{-(r-1)-a}) = \cdots = c(g_{r-1}) \right\},$$

where $c \colon \mathrm{GSp}(V,\psi) \to \mathbb{G}_m$ is the multiplicator homomorphism.

There is a natural X_M that makes (M, X_M) into a Shimura datum. Define

$$H_p := \mathrm{Stab}_{M(\mathbb{Q}_p)}(\Lambda^\S), \quad J_p := \mathrm{Stab}_{\mathrm{GSp}(V^\S_{\mathbb{Q}_p},\psi^\S_{\mathbb{Q}_p})}(\Lambda^\S).$$

Then for sufficiently small H^p, J^p, we obtain a closed immersion

$$i\colon \mathrm{Sh}_{H_pH^p}(M, X_M) \hookrightarrow \mathrm{Sh}_{J_pJ^p}(\mathrm{GSp}(V^{\S}, \psi^{\S}), S^{\S,\pm}).$$

$\mathrm{Sh}_{H_pH^p}$ has a moduli interpretation over $\mathbb{Z}_{(p)}$ (essentially: $\#\{-(r-1)-a, \dots, r-1\} = 2(r-1)+a+1$ abelian schemes up to prime-to-p isogeny endowed with certain polarizations and prime-to-p level structure).

Proposition 1.28. [Zho18, Prop. 7.2] *If J^p is sufficiently small, then i extends to a closed immersion*

$$i\colon \mathscr{S}_{H_pH^p}(M, X_M) \to \mathscr{S}_{J_pJ^p}(\mathrm{GSp}(V^{\S}, \psi^{\S}), S^{\S,\pm}).$$

Now consider the embedding $i\colon \mathcal{B}(G, \mathbb{Q}_p) \to \mathcal{B}(\mathrm{GSp}(V, \psi), \mathbb{Q}_p)$.

We still have $K_p = \mathcal{G}(\mathbb{Z}_p)$ with $\mathcal{G} = \mathcal{G}_x$, $x \in \mathcal{B}(G, \mathbb{Q}_p)$. Let $\mathfrak{g}$ be the minimal facet in $\mathcal{B}(\mathrm{GSp}(V, \psi), \mathbb{Q}_p)$ containing $i(x)$. So $\mathfrak{g}$ corresponds to some lattice chain $\Lambda^0 \supseteq \cdots \supseteq \Lambda^{r-1}$ as above. We have

$$\mathrm{Sh}_{K_pK^p}(G, X) \xrightarrow{\text{cl. imm.}} \mathrm{Sh}_{H_pH^p}(M, X_M)_{\mathbf{E}} \xrightarrow{\text{cl. imm.}} \mathrm{Sh}_{J_pJ^p}(\mathrm{GSp}(V^{\S}, \psi^{\S}), S^{\S,\pm})_{\mathbf{E}}$$

and $\mathscr{S}_K^-(G, X)$ is defined to be the closure of $\mathrm{Sh}_{K_pK^p}(G, X)$ in $\mathscr{S}_{J_pJ^p}(\mathrm{GSp}(V^{\S}, \psi^{\S}), S^{\S,\pm})_{\mathcal{O}_{\mathbf{E},(v)}}$.

Corollary 1.29. [Zho18, Cor. 7.3] *$\mathscr{S}_K^-(G, X)$ also can be described as the closure of $\mathrm{Sh}_{K_pK^p}(G, X)$ in $\mathscr{S}_{H_pH^p}(M, X_M)_{\mathcal{O}_{\mathbf{E},(v)}}$.*

Let $\mathfrak{f}'$ be a facet of $\mathcal{B}(\mathrm{GSp}(V^{\S}, \psi^{\S}), \mathbb{Q}_p)$ in the closure of $\mathfrak{g}$. Then $\mathfrak{f}'$ corresponds to a "sub-lattice chain" $\Lambda^{i_1} \supseteq \cdots \supseteq \Lambda^{i_s}$, $\{i_1, \dots, i_s\} \subseteq \{0, \dots, r-1\}$. Defining M', H_p' as above, we naturally obtain morphisms $M \to M'$ and

$$\omega_{H_pH^p, H_p'{H'}^p}\colon \mathscr{S}_{H_pH^p}(M, X_M) \to \mathscr{S}_{H_p'{H'}^p}(M', X_{M'})$$

for suitable levels H^p, ${H'}^p$ away from p.

Sketch of Proof (of Theorem 1.27): Let $\mathfrak{f}$ be the facet of K_p (that is to say $\mathfrak{f}$ is the minimal facet satisfying $x \in \mathfrak{f}$), $\mathfrak{f}'$ that of K_p'. Then $\mathfrak{f} \subseteq \overline{\mathfrak{f}'}$.

Let $x' \in \mathfrak{f}'$ be so close to x that $\mathfrak{g} \subseteq \overline{\mathfrak{g}'}$ in $\mathcal{B}(\mathrm{GSp}(V, \psi), \mathbb{Q}_p)$, where $\mathfrak{g}$ and $\mathfrak{g}'$ denote the minimal facets containing $i(x)$ and $i(x')$ respectively.

The constructions from above yield:

$$\begin{tikzcd}
\mathscr{S}^{-}_{K_pK^p}(G,X) \arrow[d, hook, "\text{closed imm.}"'] & & \mathscr{S}^{-}_{K'_p K'^p}(G,X) \arrow[d, hook, "\text{closed imm.}"] \\
\mathscr{S}_{H_p H^p}(M,X_M)_{\mathcal{O}_{\mathbf{E},(v)}} \arrow[rr, "\omega_{H_pH^p,H'_p H'^p}"] & & \mathscr{S}_{H'_p H'^p}(M',X_{M'})_{\mathcal{O}_{\mathbf{E},(v)}}
\end{tikzcd} \tag{1.30}$$

On the generic fiber we can complete this to a commutative square.

By Corollary 1.29, this implies that (1.30) also completes to a commutative square. Now normalize. □

1.7 Local structure of the integral model

1.7.1 Generizations and irreducible components

Let $\mathscr{X} \to \operatorname{Spec} \mathcal{O}_{\breve{E}}$ be a flat scheme locally of finite type; denote the special fiber by $X \to \operatorname{Spec} \bar{\mathbb{F}}_p$ and the generic fiber by $\mathcal{X} \to \operatorname{Spec} \breve{E}$. We assume that $\mathcal{X}$ is locally integral (e.g. smooth).

For example, we can consider $(\mathscr{X}, X, \mathcal{X}) = (\mathscr{S}^{-}_{K}(G,X)_{\mathcal{O}_{\breve{E}}}, \mathscr{S}^{-}_{K}(G,X)_{\mathcal{O}_{\breve{E}}} \otimes_{\mathcal{O}_{\breve{E}}} \bar{\mathbb{F}}_p, \operatorname{Sh}_K(G,X) \otimes_E \breve{E})$.

Let $\bar{x} \in X(\bar{\mathbb{F}}_p)$.

Lemma 1.31. *There is a generization x of $\bar{x}$ which lies in the generic fiber $\mathcal{X}$, and is a closed point in there, i.e., $x \in \mathcal{X}(L)$ for a finite extension $L/\breve{E}$.*

Definition 1.32. We shall call such a point x a *closed point generization* of $\bar{x}$ for short.

Proof: Due to flatness (going-down) there is *some* generization in the generic fiber; call it x_0.

By [Stacks, Tag 053U] the following set is dense (and in particular non-empty) in the closure of $\{x_0\}$ in $\mathcal{X}$:

$$\{x \in \mathscr{X} \mid x \text{ is a specialization of } x_0 \text{ and a closed point generization of } \bar{x}\}. \qquad \square$$

Lemma 1.33. *Notation as in the preceding lemma.*

The specialization $x \rightsquigarrow \bar{x}$ can be realized by an $\mathcal{O}_L$-valued point of $\mathscr{X}$.

Proof: First off, by [EGA2, 7.1.9], it can be realized by a morphism $\operatorname{Spec} R = \{\eta, s\} \to \mathscr{X}$ of $\mathcal{O}_{\breve{E}}$-schemes, where R is a discrete valuation ring such that $L \cong \kappa(\eta) = \operatorname{Quot}(R)$ as field extensions of $\kappa(x)$.

We hence get local homomorphisms of local rings $\mathcal{O}_{\breve{E}} \to \mathcal{O}_{\mathscr{X},\bar{x}} \to R$.

Thus the discrete valuation on L defined by R extends the discrete valuation on $\breve{E}$. But there is but one such extension and its valuation ring is $\mathcal{O}_L$ (by definition). □

Lemma 1.34. *Mapping x to the unique irreducible component of $\mathscr{X}$ that contains x establishes a surjection from the set of closed point generizations x of $\bar{x}$ to the set of irreducible components of $\mathscr{X}$ containing $\bar{x}$.*

Proof: If $x_0 \in \mathcal{X}$ is a generization of $\bar{x}$, then x_0 lies in a unique irreducible component of $\mathscr{X}$ because $\mathcal{X}$ is locally irreducible. Hence the map described above is well-defined.

Now for surjectivity: Given an irreducible component C of $\mathscr{X}$ containing $\bar{x}$, let $x_0 \in C$ be the generic point. Then x_0 must be in the generic fiber (else we would be able to find a generization in the generic fiber by going-down). Now go through the proof of Lemma 1.31 with this particular choice of x_0. □

1.7.2 Normalization and completion

For reference, we collect some facts concerning the passage to normalization and completion and in particular how it applies to integral models of Shimura varieties.

Fact 1.35. $\mathscr{S}_K^-(G,X) \to \operatorname{Spec} \mathcal{O}_{\breve{E}}$ is quasi-projective, so in particular of finite type.

Hence $\mathscr{S}_K^-(G,X)$ and $\mathscr{S}_K^-(G,X)_{\mathcal{O}_{\breve{E}}}$ are excellent.

As a consequence the normalization

$$\mathscr{S}_K(G,X)_{\mathcal{O}_{\breve{E}}} \xrightarrow{\nu} \mathscr{S}_K^-(G,X)_{\mathcal{O}_{\breve{E}}}$$

is finite. (This really is a normalization because normalization and completion behave well together in the excellent case (to get from $\mathcal{O}_{E^{\mathrm{ur}}}$ to $\mathcal{O}_{\breve{E}}$ and from $\mathcal{O}_{\mathbf{E},(v)}$ to $\mathcal{O}_E$) and because normalization commutes with base change along filtered colimits of smooth morphisms (to get from $\mathcal{O}_E$ to $\mathcal{O}_{E^{\mathrm{ur}}}$)).

We will always denote by $(\ )^\sim$ the integral closure of a ring in its total ring of fractions. $\bar{x}$ still denotes an $\bar{\mathbb{F}}_p$-valued point of $\mathscr{S}_K^-(G,X)$. Let $\nu^{-1}(\bar{x}) = \{\bar{y} = \bar{y}_1, \dots, \bar{y}_n\}$.
Let $\mathscr{S}^- := \mathscr{S}_K^-(G,X)_{\mathcal{O}_{\breve{E}}}$ and $\mathscr{S} := \mathscr{S}_K(G,X)_{\mathcal{O}_{\breve{E}}}$.

Fact 1.36. $\left(\nu_*\mathcal{O}_{\mathscr{S}_K^-(G,X)}\right)_{\bar{x}}^\wedge = \prod_{j=1}^n \widehat{\mathcal{O}}_{\mathscr{S},\bar{y}_j}$.

Fact 1.37. By [Stacks, Tag 0C3B]: $\left(\nu_*\mathcal{O}_{\mathscr{S}_K^-(G,X)}\right)_{\bar{x}} = \widetilde{\mathcal{O}}_{\mathscr{S}^-,\bar{x}}$.

Fact 1.38. By [Stacks, Tag 035P]: $\widetilde{A} = \prod_{\mathfrak{q}\in\mathrm{Min}(A)}(A/\mathfrak{q})^\sim$, if A is a reduced ring and $\#\,\mathrm{Min}(A) < \infty$.

Fact 1.39. $(\ )^\sim$ and $(\ )^\wedge$ commute in the case of excellence (see e.g. [EGA4, 7.6.1, 7.8.3.1(vii)]).

For instance, ${\mathcal{O}^\wedge_{\mathscr{S}^-,\bar{x}}}^\sim = {\mathcal{O}^\sim_{\mathscr{S}^-,\bar{x}}}^\wedge$ (on the right hand side, we have a completion of a $\mathcal{O}_{\mathscr{S}^-,\bar{x}}$-module).

Fact 1.40. In the case of excellence, the completion of a normal domain is a normal domain (see e.g. [GW10, 12.50]).

Thus we have

$$\begin{aligned}\prod_{\mathfrak{q}\in\mathrm{Min}(\widehat{\mathcal{O}}_{\mathscr{S}^-,\bar{x}})} (\widehat{\mathcal{O}}_{\mathscr{S}^-,\bar{x}}/\mathfrak{q})^\sim \cong {\widehat{\mathcal{O}}^\wedge_{\mathscr{S}^-,\bar{x}}}^\sim \cong {\widehat{\mathcal{O}}^\sim_{\mathscr{S}^-,\bar{x}}}^\wedge \\ \cong \left(\nu_*\mathcal{O}_{\mathscr{S}_K^-(G,X)}\right)_{\bar{x}}^\wedge \cong \prod_{j=1}^n \widehat{\mathcal{O}}_{\mathscr{S},\bar{y}_j}\end{aligned} \tag{1.41}$$

and the rings $\widehat{\mathcal{O}}_{\mathscr{S},\bar{y}_j}$ are normal domains. Hence we obtain a bijection

$$\mathrm{Min}(\widehat{\mathcal{O}}_{\mathscr{S}^-,\bar{x}}) \xleftrightarrow{1:1} \nu^{-1}(\bar{x})$$

such that there exists a numbering $\mathfrak{q}_1, \dots, \mathfrak{q}_n$ of the elements of $\mathrm{Min}(\widehat{\mathcal{O}}_{\mathscr{S}^-,\bar{x}})$ such that (1.41) restricts to an isomorphism

$$\widehat{\mathcal{O}}_{\mathscr{S}^-,\bar{x}}/\mathfrak{q}_j \cong \widehat{\mathcal{O}}_{\mathscr{S},\bar{y}_j}$$

(also see [EGA4, 7.6.2]).

Also:

$$\begin{aligned}
\widehat{\mathcal{O}}^{\sim}_{\mathscr{S}^-,\bar{x}}{}^{\wedge} &\cong \left(\prod_{\mathfrak{q}\in \mathrm{Min}(\mathcal{O}_{\mathscr{S}^-,\bar{x}})} (\mathcal{O}_{\mathscr{S}^-,\bar{x}}/\mathfrak{q})^{\sim} \right)^{\wedge} \\
&\cong \prod_{\mathfrak{q}\in \mathrm{Min}(\mathcal{O}_{\mathscr{S}^-,\bar{x}})} (\mathcal{O}_{\mathscr{S}^-,\bar{x}}/\mathfrak{q})^{\sim\wedge} \\
&\cong \prod_{\mathfrak{q}\in \mathrm{Min}(\mathcal{O}_{\mathscr{S}^-,\bar{x}})} (\mathcal{O}_{\mathscr{S}^-,\bar{x}}/\mathfrak{q})^{\wedge\sim} \\
&\overset{\text{Artin-Rees}}{\cong} \prod_{\mathfrak{q}\in \mathrm{Min}(\mathcal{O}_{\mathscr{S}^-,\bar{x}})} \left(\widehat{\mathcal{O}}_{\mathscr{S}^-,\bar{x}}/\widehat{\mathfrak{q}}\right)^{\sim}, \quad \widehat{\mathfrak{q}} = \mathfrak{q}\widehat{\mathcal{O}}_{\mathscr{S}^-,\bar{x}}.
\end{aligned}$$

Now by [KP15, 2.1.2, 4.2.2] for all $\mathfrak{q} \in \mathrm{Min}(\widehat{\mathcal{O}}_{\mathscr{S}^-,\bar{x}})$ we have that

$$\left(\widehat{\mathcal{O}}_{\mathscr{S}^-,\bar{x}}/\mathfrak{q}\right)^{\sim} = \widehat{\mathcal{O}}_{\mathscr{S}^-,\bar{x}}/\mathfrak{q} \cong R_{G,\bar{x},\mathfrak{q}} = R_G$$

is normal and $\widehat{\mathcal{O}}_{\mathscr{S}^-,\bar{x}}/\mathfrak{q} \cong \widehat{\mathcal{O}}_{\mathscr{S},\bar{y}}$ for an appropriate choice of $\mathfrak{q}$ (i.e., of x). (The notation R_G is from [KP15], where it is defined as the formal local ring of the local model discussed in section 1.9.)

1.8 Hodge tensors and (lack of) moduli interpretation

We discuss the moduli interpretation of Shimura varieties of Hodge type and the partial extension of this interpretation to the integral model as constructed in Section 1.5. Let (G, X) be a Shimura datum of Hodge type, let $(G, X) \hookrightarrow (\mathrm{GSp}(V), S^{\pm})$ be an embedding as in Definition 1.1 (4).

1.8.1 The story for the $\mathbb{C}$-valued points

Lemma 1.42. [Del82, Prop. 3.1] *There exist numbers $n, r_i, s_i \in \mathbb{Z}_{\geq 0}$ and tensors $t_i \in V^{\otimes r_i} \otimes (V^*)^{\otimes s_i}$, $1 \leq i \leq n$, such that G is the subgroup of $\mathrm{GSp}(V)$ fixing the t_i.*

Remark 1.43. (See [Mil05, Section 7].) For K a compact open subgroup of $G(\mathbb{A}_f)$, the set $\mathrm{Sh}_K(G, X)(\mathbb{C})$ of Definition 1.8 has the following moduli interpretation: isomorphism classes of triples $((W, h), (u_i)_{0\leq i\leq n}, \eta K)$, where

(a) (W,h) rational Hodge structure of type $(-1,0)+(0,-1)$,

(b) $\pm u_0$ polarization of (W,h) (i.e., either u_0 or $-u_0$ is a polarization),

(c) $u_i \in V^{\otimes r_i} \otimes (V^*)^{\otimes s_i}$ for $1 \le i \le n$,

(d) ηK is a K-orbit of isomorphisms $V \otimes \mathbb{A}_f \xrightarrow{\sim} W \otimes \mathbb{A}_f$, mapping ψ to a $\mathbb{A}_f^\times$-multiple of u_0, and every t_i to u_i $(i \ge 1)$,

such that there exists an isomorphism $a\colon W \xrightarrow{\sim} V$ mapping u_0 to a $\mathbb{Q}^\times$-multiple of ψ, every u_i to t_i $(i \ge 1)$, and h to an element of X (so ${}^a h := \mathrm{GL}(a) \circ h\colon \mathbb{S} \to \mathrm{GL}(V)$ is in X).

Sketch of Proof: Given $((W,h),(u_i)_{0\le i\le n}, \eta K)$ and a as above, we consider the pair $({}^a, a\circ\eta)$. By assumption, ${}^a h \in X$ and $a \circ \eta$ is a symplectic similitude fixing the t_i; hence $({}^a h, a\circ\eta) \in X \times G(\mathbb{A}_f)$. The double quotient now results precisely from the ambiguity in the choices of a and $\eta \in \eta K$. $\square$

Remark 1.44. Denote by $\mathbb{Q}(r)$ the rational Hodge structure of type $(-r,-r)$ with underlying vector space $\mathbb{Q}$, and denote the multiplier character $\mathrm{GSp}(V) \to \mathbb{G}_m$ as well as its restriction to G by c. Then for $h \in X$, $c \circ h \cong \mathbb{Q}(1)$ as rational Hodge structure, and the symplectic form gives an isomorphism $V \cong V^* \otimes \mathbb{Q}(1)$.

With this, we may also interpret a tensor $t \in V^{\otimes r} \otimes (V^*)^{\otimes s}$ as a multilinear map $V^{r+s} \to \mathbb{Q}$, and, if t is fixed by G, as a morphism $(V,h)^{\otimes(r+s)} \to \mathbb{Q}(r)$ of Hodge structures. Provided $t \neq 0$, this implies $r+s = 2r$, i.e. $r = s$.

Remark 1.45. Since $A \mapsto H_1(A(\mathbb{C}),\mathbb{Q})$ yields an equivalence between the category of complex abelian varieties up to isogeny and the category of polarizable rational Hodge structures of type $(-1,0)+(0,-1)$, we may also take $\mathrm{Sh}_K(G,X)(\mathbb{C})$ to be a moduli problem of abelian varieties with extra structure: consider isomorphism classes of triples $(A,(u_i)_{0\le i\le n}, \eta K)$, where

(a) A is a complex abelian variety up to isogeny,

(b) $\pm u_0$ is a polarization of the rational Hodge structure $V := H_1(A(\mathbb{C}),\mathbb{Q})$,

(c) $u_1,\dots,u_n$ is as in Remark 1.43,

(d) ηK is a K-orbit of $\mathbb{A}_f$-linear isomorphisms $V \otimes \mathbb{A}_f \to V_f(A) := T_f(A) \otimes_{\mathbb{Z}} \mathbb{Q} = \left(\varprojlim A(k)[n]\right) \otimes_{\mathbb{Z}} \mathbb{Q}$, mapping ψ to a $\mathbb{A}_f^\times$-multiple of u_0 and every t_i to u_i $(i \ge 1)$, such that there exists an isomorphism $a\colon H_1(A(\mathbb{C}), \mathbb{Q}) \xrightarrow{\sim} V$ mapping u_0 to a $\mathbb{Q}^\times$-multiple of ψ, u_i to t_i ($i \geq 1$) and h to an element of X.

Remark 1.46. When rephrasing the moduli problem appropriately, the Shimura variety as an $\mathbf{E}$-variety is given by such a moduli problem, see [Mil05, Section 14].

In particular, we have a universal abelian scheme $\mathcal{A} \to \mathrm{Sh}_K(G, X)$.

1.8.2 The story for the integral model

Integral models of Hodge type Shimura varieties in general seem not to afford nice straightforward moduli interpretations. Still, the moduli interpretation of the $\mathbb{C}$-valued points extends to *some* degree. First, we do have a generalization of Lemma 1.42 as follows.

Proposition 1.47. [Kis10, Prop. 1.3.2] *Let R be a discrete valuation ring of mixed characteristic, M a finite free R-module, and $\mathcal{G} \subseteq \mathrm{GL}(M)$ a closed R-flat subgroup with reductive generic fiber. Then $\mathcal{G}$ is defined by a finite collection of tensors $(s_\alpha)_\alpha \subset M^\otimes$.*

Here $M^\otimes$ is the direct sum of all the R-modules which can be formed from M by taking duals and (finite) tensor products.[9]

We return to the setting of Section 1.3; in particular we consider a Bruhat-Tits scheme $\mathcal{G} := \mathcal{G}_x$.

By the proposition just stated, we find a collection of tensors $(s_\alpha)_\alpha \subset (\Lambda^{\S}_{\mathbb{Z}_{(p)}})^\otimes$ whose pointwise stabilizer is the Zariski closure $G_{\mathbb{Z}_{(p)}}$ of G in $\mathrm{GL}(\Lambda^{\S}_{\mathbb{Z}_{(p)}})$.

Remark 1.48. We can for example consider, for every $j \in \{-(r-1)-a, \dots, r-1\}$, the projection $\mathrm{pr}_j\colon \Lambda^{\S}_{\mathbb{Z}_{(p)}} \to \Lambda^{j}_{\mathbb{Z}_{(p)}}$.

Since the G-action on $\Lambda^{\S}_{\mathbb{Z}_{(p)}} = \prod_{i=-(r-1)-a}^{r-1} \Lambda^{i}_{\mathbb{Z}_{(p)}}$ is diagonal, the pr_j are fixed, i.e., we may count the $\left(\mathrm{pr}_j\right)_j$ among the $(s_\alpha)_\alpha$.

Lemma 1.49. $G_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p = \mathcal{G}$.

Proof: First, $G_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$ is the Zariski closure $G_{\mathbb{Z}_p}$ of G in $\mathrm{GL}(\Lambda^{\S}_{\mathbb{Z}_p})$ and $G_{\mathbb{Z}_p} \otimes \mathbb{Q}_p = G$: This follows from [GW10, Lemma 14.6].

[9] Kisin in *loc. cit.* also allows for taking symmetric and exterior powers. As Deligne pointed out, this is unnecessary. [Del11]

According to [Stacks, Tag 056B], $G_{\mathbb{Z}_p}$ is the topological closure of G in $\mathrm{GL}(\Lambda^{\S}_{\mathbb{Z}_p})$ endowed with the reduced subscheme structure. The special fiber therefore is reduced as well, i.e. ($\mathbb{F}_p$ being perfect), geometrically reduced, i.e. (being a group scheme), smooth. $G_{\mathbb{Z}_p}$ is flat over $\mathbb{Z}_p$ by [GW10, Prop. 14.14]. $G_{\mathbb{Z}_p}$ obviously is of finite presentation over $\mathbb{Z}_p$. Hence, $G_{\mathbb{Z}_p}$ is smooth over $\mathbb{Z}_p$. It also is affine and is the stabilizer of $\Lambda^{\S}_{\mathbb{Z}_p}$ (or rather, $G_{\mathbb{Z}_p}(\mathbb{Z}_p) \subseteq G(\mathbb{Q}_p)$ is (and analogously for $G_{\mathbb{Z}_p}(\breve{\mathbb{Z}}_p) \subseteq G(\breve{\mathbb{Q}}_p)$)). $\square$

Remarks 1.50. (1) Obtaining an embedding of the Bruhat-Tits group scheme to which we can apply Proposition 1.47 is the main reason for altering the Hodge embedding in the way outlined in Section 1.3.

(2) We can also interpret the s_α as tensors in $((\Lambda^{\S}_{\mathbb{Z}_{(p)}})^*)^{\otimes}$ and that $G_{\mathbb{Z}_{(p)}}$ also is the Zariski closure of G in $\mathrm{GL}((\Lambda^{\S}_{\mathbb{Z}_{(p)}})^*)$ (using the contragredient representation).

Hodge tensors: generic fiber

Notation 1.51. Recall the universal abelian scheme $h\colon \mathcal{A} \to \mathrm{Sh}_K(G, X)$ from Remark 1.46.

We consider the local system $\mathcal{L}_B := R^1 h^{\mathrm{an}}_* \underline{\mathbb{Q}}$ on $\mathrm{Sh}_K(G,X)^{\mathrm{an}}_{\mathbb{C}}$ and the flat vector bundle $\mathcal{V}_{\mathrm{dR}} := R^1 h_* \Omega^\bullet$ with Gauß-Manin connection ∇.

Lemma 1.52. (See [CS17, Lemma 2.3.1].) *$\mathcal{L}_B$ can be identified with the local system of $\mathbb{Q}$-vector spaces on $\mathrm{Sh}_K(G,X)^{\mathrm{an}}$ given by the $G(\mathbb{Q})$-representation V and the $G(\mathbb{Q})$-torsor*

$$p\colon X \times (G(\mathbb{A}_f)/K) \to G(\mathbb{Q})\backslash(X \times (G(\mathbb{A}_f)/K) = \mathrm{Sh}_K(G,X)(\mathbb{C}).$$

Said torsor is isomorphic to the $G(\mathbb{Q})$-torsor $I := \underline{\mathrm{Isom}}((V, s_\alpha), (\mathcal{L}_B, s_{\alpha,B}))$ that maps an open subset $U \subseteq \mathrm{Sh}_K(\mathbb{C})$ to

$$I(U) = \{\beta\colon V \times U \xrightarrow{\sim} \mathcal{L}_B|_U \mid \beta(s_\alpha) = s_{\alpha,B}\},$$

where the $(s_{\alpha,B})_\alpha \subset \mathcal{L}_B^{\otimes}$ are the global sections corresponding to the $G(\mathbb{Q})$-invariant tensors $(s_\alpha)_\alpha$.

Remark 1.53. By de Rham's theorem,

$$\mathcal{V}^{\mathrm{an}}_{\mathrm{dR},\mathbb{C}} \cong \mathcal{L}_B \otimes_{\mathbb{Q}} \mathcal{O}_{\mathrm{Sh}_K(G,X)^{\mathrm{an}}}.$$

In particular the global sections $(s_{\alpha,B})_\alpha \subset \mathcal{L}_B^{\otimes}$ yield flat global sections $(s_{\alpha,\mathrm{dR}})_\alpha \subset (\mathcal{V}^{\mathrm{an}}_{\mathrm{dR},\mathbb{C}})^{\otimes}$. All such sections arise from sections in $\mathcal{V}^{\otimes}_{\mathrm{dR},\mathbb{C}}$ (which we will denote the same), see [Kis10, 2.2].

Remark 1.54. Let k/E be a field extension embeddable into $\mathbb{C}$, and choose an embedding $\mathbb{Q}_p \hookrightarrow \mathbb{C}$ and an E-embedding $\sigma\colon \bar{k} \hookrightarrow \mathbb{C}$. Let $x \in \mathrm{Sh}_K(G,X)(k)$. By p-adic Hodge theory, the embedding σ gives rise to isomorphisms

$$H^1_{\mathrm{dR}}(\mathcal{A}_x/k) \otimes_k \mathbb{C} \xrightarrow{\sim} H^1_B(\mathcal{A}_x(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^1_{\text{ét}}(\mathcal{A}_{x,\bar{k}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C},$$

and by proper base change $(\mathcal{L}_B)_x := x^{-1}\mathcal{L}_B \cong H^1_B(\mathcal{A}_x(\mathbb{C}), \mathbb{Q})$.

Notation 1.55. We denote by $s_{\alpha,B,x}$ the fiber of $s_{\alpha,B}$ at x, and by $s_{\alpha,\mathrm{dR},x}$ and $s_{\alpha,\text{ét},x}$, respectively, the images of $s_{\alpha,B,x}$ under the above isomorphisms. For the Betti–étale comparison, we don't need to go all the way to $\mathbb{C}$ and have $s_{\alpha,\text{ét},x} \in H^1_{\text{ét}}(\mathcal{A}_{x,\bar{k}}, \mathbb{Q}_p)$.

Remark 1.56. $(s_{\alpha,\mathrm{dR},x}, s_{\alpha,\text{ét},x})$ is an absolute Hodge cycle in the sense of [Del82], for every α.

Lemma 1.57. [Kis10, Lemma 2.2.1] *The $\mathrm{Gal}(\bar{k}/k)$-action on $H^1_{\text{ét}}(\mathcal{A}_{x,\bar{k}}, \mathbb{Q}_p)$ fixes every $s_{\alpha,\text{ét},x}$ and factors through $G(\mathbb{Q}_p)$. Moreover, $s_{\alpha,\mathrm{dR},x} \in H^1_{\mathrm{dR}}(\mathcal{A}_x/k)$.*

In particular, $(s_{\alpha,\mathrm{dR},x}, s_{\alpha,\text{ét},x})$ is independent of the choices made above.

Corollary 1.58. [Kis10, Lemma 2.2.2] *$s_{\alpha,\mathrm{dR}}$ is defined over E for all α, i.e., $s_{\alpha,\mathrm{dR}} \in \mathcal{V}^{\otimes}_{\mathrm{dR},E}$.*

Hodge tensors: special fiber

Notation 1.59. Denote by $\mathscr{A}_{\text{Siegel}}$ the universal abelian scheme over $\mathscr{S}_J(\mathrm{GSp}(V^{\S}), S^{\S,\pm})$, cf. Remark 1.23, and by $\mathscr{A}$ the pullback of $\mathscr{A}_{\text{Siegel}}$ to $\mathscr{S}_K(G,X)$. Then the pullback of $\mathscr{A}$ to $\mathrm{Sh}_K(G,X)_E$ agrees with the pullback of $\mathcal{A}$ of Remark 1.46.

We will occasionally call $\mathscr{A}$ the *"universal" abelian scheme* (with quotation marks).

Let $\bar{x} \in \mathscr{S}_K(G,X)(\bar{\mathbb{F}}_p)$ and let $\mathbb{D}_{\bar{x}} := \mathbb{D}(\mathscr{A}_{\bar{x}}[p^\infty])(W)$ be the Dieudonné module of the associated fiber of the "universal" abelian scheme, $W = W(\bar{\mathbb{F}}_p) = \breve{\mathbb{Z}}_p$. Choose a closed point generization $x \in \mathrm{Sh}_K(G,X)(L)$, L/E finite field extension, in the sense of Definition 1.32. Note that L can be embedded into $\mathbb{C}_p \cong \mathbb{C}$. Then

$$H^1_{\text{ét}}(\mathscr{A}_{x,\bar{E}}, \mathbb{Z}_p) \cong (T_p\mathscr{A}_{x,\bar{E}})^* \cong (T_p\mathscr{A}^*_{x,\bar{E}})(-1) \cong (\Lambda^{\S})^*,$$

allowing us to identify the tensors s_α with tensors $s_{\alpha,\text{ét},x} \in H^1(\mathscr{A}_{x,\bar{E}}, \mathbb{Z}_p)^{\otimes}$. Again, these tensors can be shown to be $\mathrm{Gal}(\bar{E}/L)$-invariant. Now we have the p-adic comparison isomorphism

$$H^1_{\text{ét}}(A_{x,\bar{E}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{cris}} \cong H^1_{\text{cris}}(\mathscr{A}_{\bar{x}}/W) \otimes_W B_{\text{cris}} = \mathbb{D}(\mathscr{A}_{\bar{x}}[p^\infty])(W) \otimes_W B_{\text{cris}}, \quad (1.60)$$

and by [KP15, 3.3.8], via this isomorphism, the $s_{\alpha,\text{ét},x}$ also correspond to tensors $s_{\alpha,0} := s_{\alpha,0,\bar{x}}$ in $\mathbb{D}_{\bar{x}}^{\otimes}$. In fact, we get an isomorphism

$$(\Lambda^{\S})^* \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p \cong \mathbb{D}_{\bar{x}} \tag{1.61}$$

identifying $s_\alpha \otimes 1$ with $s_{\alpha,0,\bar{x}}$.

Globalizing crystalline tensors

Above we constructed crystalline tensors fiberwise. This will not suffice to understand the local geometry of the special fiber of the integral model. Therefore we now “globalize” the tensors.

Let $\bar{x} \in \mathscr{S}_K(G,X)(\bar{\mathbb{F}}_p)$. As in Section 1.7.2, we write $R_G := R_{G,\bar{x}} := \widehat{\mathcal{O}}_{\mathscr{S}_K(G,X)_{\mathcal{O}_{\breve{E}}},\bar{x}}$. (So $\operatorname{Spf} R_G$ is the $\bar{x}$-adic completion of $\mathscr{S}_K(G,X)_{\mathcal{O}_{\breve{E}}}$.)

Remark 1.62. In [KP15, 3.2.12, 3.2.14] a construction of a “universal” deformation p-divisible group

$$\mathscr{G}_{R_G} \to \operatorname{Spec} R_G \tag{1.63}$$

is given, characterized by its Dieudonné display, to wit $\operatorname{DDisp}(\mathscr{G}_{R_G}) = \mathbb{D}_{\bar{x}} \otimes_W \hat{W}(R_G)$ endowed with a natural Dieudonné display structure.

$\mathscr{G}_{R_G}$ can be identified with the pullback of the p-divisible group of the “universal” abelian scheme $\mathscr{A}$.

Remark 1.64. R_G is a complete local normal noetherian ring with perfect residue field.

Remark 1.65. By [Lau14, Theorem B], the Dieudonné crystal associated with $\operatorname{DDisp}(\mathscr{G}_{R_G})$ coincides with the Dieudonné crystal $\mathbb{D}(\mathscr{G}_{R_G})$ of $\mathscr{G}_{R_G}$.

In fact, $\operatorname{DDisp}(\mathscr{G}_{R_G}) = \mathbb{D}(\mathscr{G}_{R_G})^\vee(\hat{W}(R_G))$ endowed with a natural Dieudonné display structure, cf. [KP15, 3.1.7].

In particular we get tensors $t_{\alpha,\bar{x}}^{\mathrm{def}} \in \mathbb{D}(\mathscr{G}_{R_G})^\vee(\hat{W}(R_G))^{\otimes}$ corresponding to $s_{\alpha,0,\bar{x}} \otimes 1 \in (\mathbb{D}_{\bar{x}} \otimes_W \hat{W}(R_G))^{\otimes}$, and accordingly tensors $t_{\alpha,\bar{x}}^{\mathrm{def}} \otimes 1 \in \mathbb{D}(\mathscr{G}_{R_G})^\vee(W(R_G))^{\otimes}$.

Definition 1.66. (See [Ham17, Definition 2.8].) Let $\mathscr{G} \to S$ be a p-divisible group over a formally smooth $\mathbb{F}_p$-scheme S. Denote by $\mathbb{D}(\mathscr{G})$ its contravariant Dieudonné crystal of $\mathscr{G}$ [BBM82, Def. 3.3.6], and by $\mathbb{1} := \mathbb{D}(\underline{\mathbb{Q}_p/\mathbb{Z}_p}_S)$ the unit object in the tensor category of locally free $\mathcal{O}_{S/\mathbb{Z}_p,\mathrm{CRIS}}$-modules. Note that $\mathbb{D}(\mathscr{G})$ comes with a Frobenius morphism $\mathbb{D}(\mathscr{G})^{(p)} \to \mathbb{D}(\mathscr{G})$, and similar for $\mathbb{1}$.

(1) A *tensor* t of $\mathbb{D}(\mathscr{G})$ is a morphism $\mathbb{1} \to \mathbb{D}(\mathscr{G})^{\otimes}$ of locally free $\mathcal{O}_{S/\mathbb{Z}_p,\mathrm{CRIS}}$-modules.[10]

(2) A tensor t of $\mathbb{D}(\mathscr{G})$ is called a *crystalline Tate tensor on* $\mathscr{G}$ if it induces a morphism of F-isocrystals $\mathbb{1} \to \mathbb{D}(\mathscr{G})[\frac{1}{p}]^{\otimes}$.

Remarks 1.67. (1) To give a tensor as in the preceding definition therefore means the following: For every (U, T, i, δ) with

(a) an S-scheme U,

(b) a $\mathbb{Z}_p$-scheme T on which p is locally nilpotent,

(c) a closed $\mathbb{Z}_p$-immersion $i\colon U \hookrightarrow T$,

(d) a pd-structure δ on the ideal in $\mathcal{O}_T$ defining the immersion i, compatible with the canonical pd-structure on $p\mathbb{Z}_p$,

functorially to give a morphism $\mathbb{1}(U, T, i, \delta) = \mathcal{O}_T(T) \to \mathbb{D}(\mathscr{G})(U, T, i, \delta)^{\otimes}$ of $\mathcal{O}_T(T)$-modules, i.e., functorially to give elements of $\mathbb{D}(\mathscr{G})(U, T, i, \delta)^{\otimes}$.

(2) Assume $S = \operatorname{Spec} A$, where A is an $\mathbb{F}_p$-algebra which has a p-basis (e.g., A perfect) *or* which satisfies [Jon95, (1.3.1.1)], the latter signifying the existence of an ideal $I \subseteq A$ such that

- A is noetherian and I-adically complete,
- A is formally smooth as a topological $\mathbb{F}_p$-algebra with the I-adic topology,
- A/I contains a field with a finite p-basis and is finitely generated as an algebra over this field.

Also fix a *lift* $\tilde{A}$ of A in the sense of [Jon95, Def. 1.2.1], i.e., a p-adically complete $\mathbb{Z}_p$-flat ring $\tilde{A}$ together with an isomorphism $\tilde{A}/p\tilde{A} \cong A$ and a ring endomorphism $\sigma\colon \tilde{A} \to \tilde{A}$ such that $\sigma(a) \equiv a^p \mod p\tilde{A}$. Note that if A is perfect, then $\tilde{A} := W(A)$ with the usual Frobenius lift works.

Then by [BM90, Prop. 1.3.3] and [Jon95, Cor. 2.2.3] the category of crystals of quasi-coherent $\mathcal{O}_{\operatorname{Spec}(A)/\mathbb{Z}_p,\mathrm{CRIS}}$-modules is equivalent to the category of p-adically complete $\tilde{A}$-modules M endowed with an integrable topologically quasi-nilpotent connection $\nabla\colon M \to M \otimes_{\tilde{A}} \hat{\Omega}^1_{\tilde{A}}$.

[10] $(\ )^{\otimes}$ here (and also in point (2)) is defined as it is defined for R-modules in Prop. 1.47.

(3) In the setting of (2), say (M, ∇) is the image of $\mathbb{D}(\mathscr{G})$ under the equivalence. Then $M = \mathbb{D}(\mathscr{G})(\tilde{A})$, and to give a tensor $\mathbb{1} \to \mathbb{D}(\mathscr{G})^{\otimes}$ means to give a horizontal section of $M^{\otimes}$.

Proposition 1.68. [HK17, Prop. 3.3.1, Cor. 3.3.7] *Denote by $\mathscr{G}_{\overline{\mathscr{S}}^{\mathrm{perf}}}$ and $\mathscr{G}_{\hat{\mathscr{S}}}$ the pullbacks of the p-divisible group $\mathscr{A}[p^\infty]$ to the perfection*[11] $\overline{\mathscr{S}}^{\mathrm{perf}} := (\mathscr{S}_K(G,X) \otimes_{\mathcal{O}_E} \bar{\mathbb{F}}_p)^{\mathrm{perf}}$ *and the p-adic completion $\hat{\mathscr{S}}$ of $\mathscr{S}_K(G,X) \otimes_{\mathcal{O}_E} \mathcal{O}_{\breve{E}}$, respectively.*

For a p-divisible group $\mathscr{X}$ over a p-adic formal scheme $\mathfrak{S}$ denote by $P(\mathscr{X})$ the locally free $W(\mathcal{O}_{\mathfrak{S}})$-module given by $P(\mathscr{X})(\operatorname{Spf} A) = \mathbb{D}(\mathscr{X}_A)^\vee(W(A))$ for every open affine formal subscheme $\operatorname{Spf} A \subseteq \mathfrak{S}$.

Then there exist tensors $(t_\alpha)_\alpha \subset P(\mathscr{G}_{\hat{\mathscr{S}}})^{\otimes}$ whose pullback to $P(\mathscr{G}_{R_G,\bar{x}})^{\otimes}$ coincides with $t^{\mathrm{def}}_{\alpha,\bar{x}} \otimes 1$ for all $\bar{x} \in \mathscr{S}_K(G,X)(\bar{\mathbb{F}}_p)$.

Via pullback, these t_α yield crystalline Tate tensors $(\bar{t}_\alpha)_\alpha$ on $\mathscr{G}_{\overline{\mathscr{S}}^{\mathrm{perf}}}$.

Moreover, if $x \in \mathscr{S}_K(G,X)(\mathcal{O}_L)$, $L/\breve{E}$ finite field extension, is a closed point generization of $\bar{x} \in \mathscr{S}_K(G,X)(\bar{\mathbb{F}}_p)$, then $s_{\alpha,\text{ét},x} \in (T_p\mathscr{A}_{x,\bar{E}})^{\otimes}$ gets identified with $\bar{t}_{\alpha,\bar{x}} \in \mathbb{D}(\mathscr{A}_{\bar{x}}[p^\infty])(W(\bar{\mathbb{F}}_p))$ by the p-adic comparison isomorphism (1.60).

1.9 The local model

To give a very rough idea of what the *local model* to be discussed in this section is supposed to accomplish: It should be an $\mathcal{O}_E$-scheme that is étale-locally isomorphic to $\mathscr{S}_K(G,X)$, but easier to understand by virtue of being of a more “linear-algebraic flavor”. In actuality however, the theory of local models quickly gets quite complicated once one departs from the simplest examples.

1.9.1 The Siegel case

We do start with the simplest example.

We consider the standard Iwahori subgroup $I_p \subseteq \mathrm{GSp}_{2g}(\mathbb{Z}_p)$, defined as the preimage of the standard Borel subgroup of $\mathrm{GSp}_{2g}(\mathbb{F}_p)$. In terms of the building (cf. Remark 1.14), it corresponds to the lattice chain $\mathcal{L}_{\mathrm{full}}$ given by

$$\begin{aligned} \Lambda^0 = \mathbb{Z}_p^{2g} \supsetneqq \Lambda^1 = \mathbb{Z}_p^{2g-1} \oplus p\mathbb{Z}_p \supsetneqq \Lambda^2 = \mathbb{Z}_p^{2g-2} \oplus p\mathbb{Z}_p^2 \\ \supsetneqq \cdots \supsetneqq \Lambda^{2g-1} = \mathbb{Z}_p \oplus p\mathbb{Z}_p^{2g-1} \supsetneqq p\Lambda^0 = p\mathbb{Z}_p^{2g} \end{aligned} \tag{1.69}$$

[11] This being the inverse perfection of $\mathscr{S}_K(G,X) \otimes_{\mathcal{O}_E} \bar{\mathbb{F}}_p$ in the terminology of [BG18, Section 5].

of period length $2g$.

Consider a subset $J = \{j_0 > \cdots > j_{m-1}\} \subseteq \{1, \ldots, 2g\}$ such that for each $j \in J$ with $1 \leq j \leq 2g - 1$ also $2g - j \in J$, and let K_p be the parahoric subgroup associated with the partial lattice chain $\mathcal{L} \subseteq \mathcal{L}_{\text{full}}$ obtained from $\{\Lambda^j \mid j \in J\}$.

Define a scheme $\tilde{\mathscr{S}}_K(G,X)$ over $\mathscr{S}_K(G,X)$ as follows:

$$\tilde{\mathscr{S}}_K(G,X)(S) = \left\{ (A, \bar{\lambda}, \eta^p, \tau) \;\middle|\; \begin{array}{l} (A, \bar{\lambda}, \eta^p) \in \mathscr{S}_K(\mathrm{GSp}_{2g}, S^{\pm})(S), \\ \tau \colon H^1_{\mathrm{dR}}(A) \xrightarrow{\sim} \mathcal{L} \otimes \mathcal{O}_S \text{ isomorphism of lattice chains} \end{array} \right\}$$

for every $\mathbb{Z}_p$-scheme S.

By [RZ96, Appendix to Chap. 3], $\tilde{\mathscr{S}}_K(G,X) \to \mathscr{S}_K(G,X)$ is a Zariski torsor under the automorphism group of $\mathcal{L}$, i.e., the Iwahori group scheme.

This motivates the definition of the local model $M^{\mathrm{loc}}_{K_p} \to \operatorname{Spec} \mathbb{Z}_p$ as the "moduli space of Hodge filtrations"; more precisely:

Remark 1.70. (See [Gör03, p. 91].) $M^{\mathrm{loc}}_{K_p}(S)$ is the set of isomorphism classes of commutative diagrams

$$\begin{tikzcd}
\Lambda_S^{j_0} \arrow[r] & \Lambda_S^{j_1} \arrow[r] & \cdots \arrow[r] & \Lambda_S^{j_0} \arrow[r, "\cdot p"] & \Lambda_S^{j_{m-1}} \\
\mathcal{F}^{j_0} \arrow[r] \arrow[u, hook] & \mathcal{F}^{j_1} \arrow[r] \arrow[u, hook] & \cdots \arrow[r] & \mathcal{F}^{j_0} \arrow[r] \arrow[u, hook] & \mathcal{F}^{j_{m-1}} \arrow[u, hook]
\end{tikzcd}$$

with $\Lambda_S^j := \Lambda^j \otimes_{\mathbb{Z}_p} \mathcal{O}_S$, $\mathcal{F}^j \subseteq \Lambda_S^j$ locally direct summand of rank g, such that for all $j \in J$, $\mathcal{F}^j \to \Lambda_S^j \overset{\psi}{\cong} (\Lambda_S^{2g-j})^* \to (\mathcal{F}^{2g-j})^*$ vanishes, ψ being the symplectic pairing.

By Grothendieck-Messing theory, one obtains a diagram

$$\begin{tikzcd}
& \tilde{\mathscr{S}}_K(G,X) \arrow[dl, "\mathrm{Aut}(\mathcal{L})\text{-torsor}"'] \arrow[dr, "\text{smooth of rel. dim. } \dim \mathrm{Aut}(\mathcal{L})"] & \\
\mathscr{S}_K(G,X) & & M^{\mathrm{loc}}_K
\end{tikzcd}$$

Since both morphisms in this diagram are smooth of the same dimension, it follows that for every finite field extension $\mathbb{F}_q/\mathbb{F}_p$ and every point $x \in \mathscr{S}_K(G,X)(\mathbb{F}_q)$, there exists a point $y \in M_K^{\mathrm{loc}}(\mathbb{F}_q)$ and an isomorphism $\mathcal{O}^h_{\mathscr{S}_K(G,X),x} \cong \mathcal{O}^h_{M_K^{\mathrm{loc}},y}$ of henselizations.

In many (P)EL situations one has similar descriptions with the obvious extra structures. Sometimes however the so-called "naive" local models so obtained additionally need to be flattened, which leaves one without any self-evident moduli interpretation.

1.9.2 The relation between the integral and the local model

Generalizing the Siegel example, we axiomatically characterize the relationship between the integral model of the Shimura variety and its local model: One wants a *local model diagram*, i.e., a diagram of $\mathcal{O}_E$-schemes functorial in K

$$\begin{tikzcd} & \tilde{\mathscr{S}}_K(G,X) \arrow[dl, "\mathcal{G}_{\mathcal{O}_E}\text{-torsor}"'] \arrow[dr, "\text{equivariant and smooth of rel. dim. } \dim \mathcal{G}_{\mathcal{O}_E}"] & \\ \mathscr{S}_K(G,X) & & M_K^{\mathrm{loc}} \end{tikzcd} \tag{1.71}$$

where M_K^{loc} is a projective flat $\mathcal{O}_E$-scheme with an action of $\mathcal{G} \otimes_{\mathbb{Z}_p} \mathcal{O}_E$ and generic fiber the canonical model of $G_{\bar{\mathbb{Q}}_p}/P_{\mu^{-1}}$ over E.

By Kisin-Pappas [KP15] we do actually have such a diagram in our situation.

1.9.3 The Pappas-Zhu construction

In [PZ13], Pappas and Zhu give a construction of the local model in quite a general context, in particular with no assumptions going beyond our running assumptions 1.18.

Remark 1.72. To this end, they construct an affine smooth group scheme $\underline{\mathcal{G}}_K \to \mathbb{A}^1_{\mathbb{Z}_p} = \operatorname{Spec} \mathbb{Z}_p[t]$ with the following key properties:

(1) $\underline{\mathcal{G}}_K$ has connected fibers,

(2) $\underline{\mathcal{G}}_K$ is reductive over $\operatorname{Spec} \mathbb{Z}_p[t^{\pm 1}]$,

(3) $\underline{\mathcal{G}}_K \otimes_{\mathbb{Z}_p[t], t\mapsto p} \mathbb{Z}_p \cong \mathcal{G}_K$, in particular

- $\underline{\mathcal{G}}_K \otimes_{\mathbb{Z}_p[t],t\mapsto p} \mathbb{Q}_p \cong G_{\mathbb{Q}_p}$ and
- $\underline{\mathcal{G}}_K \otimes_{\mathbb{Z}_p[t]} \mathbb{F}_p := \underline{\mathcal{G}}_K \otimes_{\mathbb{Z}_p[t],t\mapsto 0} \mathbb{F}_p \cong \mathcal{G}_K \otimes \mathbb{F}_p$,

(4) $\underline{\mathcal{G}}_K \otimes_{\mathbb{Z}_p[t]} \mathbb{Q}_p[[t]]$ is parahoric for $\underline{\mathcal{G}}_K \otimes_{\mathbb{Z}_p[t]} \mathbb{Q}_p((t))$,

(5) $\underline{\mathcal{G}}_K \otimes_{\mathbb{Z}_p[t]} \mathbb{F}_p[[t]]$ is parahoric for $\underline{\mathcal{G}}_K \otimes_{\mathbb{Z}_p[t]} \mathbb{F}_p((t))$.

Definition and Remark 1.73. Let X_μ be the canonical model of $G_{\bar{\mathbb{Q}}_p}/P_{\mu^{-1}}$ over E, where for a cocharacter ν one defines $P_\nu := \{g \in G \mid \lim_{t\to 0} \nu(t) g \nu(t)^{-1} \text{ exists}\}$.

Let S_μ be the closed subvariety of $\mathrm{Gr}_G \times_{\mathbb{Q}_p} E$ with

$$S_\mu(\bar{\mathbb{Q}}_p) = G(\bar{\mathbb{Q}}_p[[t]])\mu(t)G(\bar{\mathbb{Q}}_p[[t]])/G(\bar{\mathbb{Q}}_p[[t]]).$$

Then S_μ can be G_E-equivariantly identified with X_μ.

Definition 1.74. The local model $M^{\mathrm{loc}}_{G,\mu,K}$ now is defined to be the Zariski closure of $X_\mu \subseteq \mathrm{Gr}_G \times_{\mathbb{Q}_p} E$ in $\mathrm{Gr}_{\underline{\mathcal{G}}_K,\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathcal{O}_E$, where $\mathrm{Gr}_{\underline{\mathcal{G}}_K,\mathbb{Z}_p} := \mathrm{Gr}_{\underline{\mathcal{G}}_K,\mathbb{A}^1_{\mathbb{Z}_p}} \otimes_{\mathbb{A}^1_{\mathbb{Z}_p}, u\mapsto p} \mathbb{Z}_p$ is a base change of the global affine Graßmannian as defined in [PZ13].

2 Central leaves in the case of parahoric reduction

We begin by giving some history. The foliation given by the central leaves was introduced by Oort [Oor04; Oor09] in the setting of a p-divisible group over a characteristic p scheme (cf. Remark 2.2 below). As already indicated in the introduction, the idea is to consider for every isomorphism class of p-divisible groups over an algebraically closed field of characteristic p the locus where the isomorphism class of the geometric fibers of the p-divisible group is the given one. It is not at all readily apparent why this would be a reasonable notion in geometrical or topological terms. The key tool here is given by the slope filtrations introduced by Zink [Zin01b], and a key insight is that there is a number $N \geq 0$ such that a p-divisible group over an algebraically closed field of characteristic p is in the prescribed isomorphism class if and only if the analogous statement holds for the p^N-torsion subgroups. For Shimura varieties of Hodge type and good reduction, the central leaves were studied by Mantovan [Man04; Man05] (PEL cases), Vasiu [Vas08] and Zhang [Zha15] (a good survey), among others.

He and Rapoport formulated a set of axioms that integral models for Shimura varieties with parahoric level are supposed to satisfy in order for them to merit the label "canonical". Among these is having a notion of well-behaved central leaves. This is what we investigate in the Hodge type situation. As already mentioned, Zhou [Zho18] concurrently and independently also worked on this and obtained very similar results in a different way, and we use some of his results from the first version of the preprint, which didn't yet address the change-of-parahoric map between central leaves. Central leaves in the parahoric level case also are subject in work of Hamacher and Kim [Ham17; Kim19; HK17], but they do not deal with changing the parahoric level.

We still fix a Shimura datum (G, X) of Hodge type, a parahoric subgroup $K_p \subseteq G(\mathbb{Q}_p)$ (associated with a Bruhat-Tits group scheme $\mathcal{G} \to \operatorname{Spec} \mathbb{Z}_p$) and a sufficiently small open compact subgroup $K^p \subseteq G(\mathbb{A}_f^p)$. We also keep up our standard assumptions 1.18.

2.1 Definition of central leaves

We use the notation for Hodge tensors, in particular $(s_\alpha)_\alpha$ and $(s_{\alpha,0})_\alpha$, established in Section 1.8.2.

Definition 2.1. Two points $\bar{x}_1, \bar{x}_2 \in \mathscr{S}_K(G,X)(\bar{\mathbb{F}}_p)$ lie in the same *central leaf* if there is an isomorphism $\mathbb{D}_{\bar{x}_1} \cong \mathbb{D}_{\bar{x}_2}$ of Dieudonné modules that takes $s_{\alpha,0,\bar{x}_1}$ to $s_{\alpha,0,\bar{x}_2}$.

They lie in the same *naive central leaf* if there is an arbitrary isomorphism $\mathbb{D}_{\bar{x}_1} \cong \mathbb{D}_{\bar{x}_2}$ of Dieudonné modules (i.e., an isomorphism $\mathscr{A}_{\bar{x}_1}[p^\infty] \cong \mathscr{A}_{\bar{x}_1}[p^\infty]$ of p-divisible groups).

Remark 2.2. More generally, one can make analogous definitions when given any abelian scheme $\mathcal{A} \to S$ over any $\mathbb{F}_p$-scheme S (such that for every point we are given tensors on the Dieudonné module of the corresponding abelian variety). Given two arbitrary points, one compares them after going to a common algebraically closed extension of the residue fields, cf. [Oor04].

Notation 2.3. Recall that we denote the completion of the maximal unramified extension $\mathbb{Q}_p^{\mathrm{ur}}$ of $\mathbb{Q}_p$ by $\breve{\mathbb{Q}}_p$, and its ring of integers accordingly by $\breve{\mathbb{Z}}_p$. We set $\breve{K} := \breve{K}_p := \mathcal{G}(\breve{\mathbb{Z}}_p)$ and define $\breve{K}_\sigma$ to be the graph of the Frobenius $\sigma\colon \breve{K} \to \breve{K}$. So dividing out the action of $\breve{K}_\sigma$ (which is mostly how $\breve{K}_\sigma$ will make an appearance) means dividing out the action of $\breve{K}$ by σ-conjugation.

Definition 2.4. The central leaves are the fibers of the map

$$\Upsilon = \Upsilon_K \colon \mathscr{S}_K(G,X)(\bar{\mathbb{F}}_p) \to G(\breve{\mathbb{Q}}_p)/\breve{K}_\sigma$$

given as follows: For $\bar{x} \in \mathscr{S}_K(G,X)(\bar{\mathbb{F}}_p)$ there is an isomorphism $\beta\colon V^*_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p \cong \mathbb{D}_{\bar{x}}$ (equation (1.61)) sending $s_\alpha \otimes 1$ to $s_{\alpha,0,\bar{x}}$. We hence can interpret the Frobenius on $\mathbb{D}_{\bar{x}}$ as an element of $G(\breve{\mathbb{Q}}_p)/\breve{K}_\sigma$, where dividing out $\breve{K}_\sigma$ rids us of the ambiguity introduced by the choice of the isomorphism β.

Notation 2.5. We write

$$\begin{aligned} G(\breve{\mathbb{Q}}_p) \to C(G) &:= G(\breve{\mathbb{Q}}_p)/\breve{K}_\sigma \to B(G) := G(\breve{\mathbb{Q}}_p)/G(\breve{\mathbb{Q}}_p)_\sigma, \\ b \mapsto \quad & [\![b]\!] \quad \mapsto [b]. \end{aligned}$$

2.1.1 An alternative characterization of the central leaves

Consider two points $\bar{x}_1, \bar{x}_2 \in \mathscr{S}_K(G,X)(\bar{\mathbb{F}}_p)$ in the same central leaf, i.e., such that there is an isomorphism $\mathbb{D}_{\bar{x}_1} \cong \mathbb{D}_{\bar{x}_2}$ of Dieudonné modules that takes $s_{\alpha,0,\bar{x}_1}$ to $s_{\alpha,0,\bar{x}_2}$.

$\bar{x}_j$ for $j = 1, 2$ has an associated isogeny chain of abelian schemes in $\mathscr{S}_N(\mathrm{GSp}(V), S^{\pm})$, cf. Remark 1.21, and $\mathbb{D}_{\bar{x}_j} = \mathbb{D}(\prod_{i=-(r-1)-a}^{r-1} A_{j,i}) = \prod_{i=-(r-1)-a}^{r-1} \mathbb{D}(A_{j,i})$, where $(A_{j,i})_i$ is the isogeny chain associated with $\bar{x}_j$ under $\mathscr{S}_K(G,X) \to \mathscr{S}_J(\mathrm{GSp}(V^\S), S^{\S,\pm})$, cf. Remark 1.23 and Proposition 1.26. Note that $A_j = \prod_{i=-(r-1)-a}^{r-1} A_{j,i}$ (plus extra structure) is the point associated with $\bar{x}_j$ under $\mathscr{S}_K(G,X) \to \mathscr{S}_J(\mathrm{GSp}(V^\S), S^{\S,\pm})$.

Lemma 2.6. *Any tensor-preserving isomorphism $\mathbb{D}_{\bar{x}_1} \cong \mathbb{D}_{\bar{x}_2}$ yields, for all i, isomorphisms $\mathbb{D}(A_{1,i}) \cong \mathbb{D}(A_{2,i})$.*

Proof: This follows immediately from Remark 1.48. □

We may thus rephrase our definition of central leaves as follows.

Lemma 2.7. *Let $\bar{x}_1, \bar{x}_2 \in \mathscr{S}_K(G,X)(\bar{\mathbb{F}}_p)$ and, as above, denote by $A_{j,i}$ the abelian varieties in the associated isogeny chains, $j = 1, 2$.*

$\bar{x}_1$ and $\bar{x}_2$ lie in the same central leaf if and only if there is an isomorphism between the associated rational Dieudonné modules of $A_{j,i}$ (which is common to all $A_{j,i}$, j fixed, i variable) respecting the tensors and identifying the lattices $\mathbb{D}(A_{1,i})$ and $\mathbb{D}(A_{2,i})$ for all i.

"Respecting the tensors" means the following: We get an induced identification of the rational Dieudonné modules of A_1 and A_2. This identification has to preserve the tensors.

2.2 Local closedness of central leaves

2.2.1 Topological lemmas

We begin with some purely topological preliminaries which we shall use to globalize statements about formal neighborhoods.

Lemma 2.8. *Let X be a topological space with a subset $A \subseteq X$, and for $x \in X$ let $U_x \subseteq X$ be the set of generizations of x. Consider the statements*

(i) $A \cap U_x \subseteq U_x$ closed (resp. open) for all $x \in X$.

(ii) $A \cap U_x \subseteq U_x$ closed (resp. open) for all closed points $x \in X$.

(iii) $A \subseteq X$ stable under specialization (resp. generization).

We have (i) $\implies$ (ii), (iii)*, and if every $x \in X$ has a specialization in x that is a closed point, we also have* (ii) $\implies$ (iii).

PROOF: Denote the closure of A in X by $\mathrm{cl}_X(A)$.

For "(i) $\implies$ (iii)" (with "closed" and "specialization", respectively) let $x \in A$ and $s \in \mathrm{cl}_X(\{x\})$. Then $s, x \in U_s$, and $s \in \mathrm{cl}_X(\{x\}) \cap U_s = \mathrm{cl}_{U_s}(\{x\})$. By assumption $A \cap U_s \subseteq U_s$ is stable under specialization, hence $s \in A \cap U_s \subseteq A$.

The rest is proven along similar lines. □

Example 2.9. $\mathbb{N} \subseteq \mathbb{A}^1_{\mathbb{C}}$ satisfies property (i) from the lemma but is not closed (hence not constructible).

Lemma 2.10. *Let X be a sober topological space and let $A \subseteq X$ be stable under both generization and specialization.*

Then A is a union of irreducible components of X.

If additionally X only has finitely many irreducible components, A even is a union of connected components and is open and closed.

PROOF: Consider the unique generic points $\{\eta_i\}_{i \in I}$ of the irreducible components of X. By assumption, we have, firstly, that if A contains η_i then it contains the entire irreducible component $\overline{\{\eta_i\}}$, and, secondly, that if A contains any one point of $\overline{\{\eta_i\}}$, then it also contains η_i. Similarly, if A contains any one irreducible component $\overline{\{\eta_i\}}$, it also contains all irreducible components that meet $\overline{\{\eta_i\}}$. □

2.2.2 Local closedness

To show the local closedness of central leaves, we can content ourselves with a construction in the spirit of Proposition 1.68 but somewhat simpler. Namely, we pull back (1.63) to obtain

$$\begin{aligned}\mathscr{G}_{\bar{x}} \to \operatorname{Spec} \bar{R}_G &:= \operatorname{Spec} R_G \otimes_{\mathcal{O}_{\breve{E}}} \bar{\mathbb{F}}_p \\ &= \operatorname{Spec}(\widehat{\mathcal{O}}_{\mathscr{S}_K(G,X)_{\mathcal{O}_{\breve{E}}},\bar{x}} \otimes \bar{\mathbb{F}}_p) = \operatorname{Spec}(\widehat{\mathcal{O}}_{\mathscr{S}_K(G,X) \otimes_{\mathcal{O}_{\mathbf{E},(v)}} \bar{\mathbb{F}}_p, \bar{x}}).\end{aligned}$$

The Dieudonné display of $\mathscr{G}_{\bar{x}}$ then is

$$\mathrm{DDisp}(\mathscr{G}_{\bar{x}}) = \mathbb{D}(\mathscr{G}_{\bar{x}})^\vee(\hat{W}(\bar{R}_G)) = \mathbb{D}_{\bar{x}} \otimes_{\breve{\mathbb{Z}}_p} \hat{W}(\widehat{\mathcal{O}}_{\mathscr{S}_K(G,X) \otimes_{\mathcal{O}_{\mathbf{E},(v)}} \bar{\mathbb{F}}_p, \bar{x}}).$$

In particular we can consider the tensors $(s_{\alpha,0} \otimes 1)_\alpha$ on this Dieudonné display and on $\mathbb{D}(\mathscr{G}_{\bar{x}})^\vee(W(\bar{R}_G))$.

Thus we get crystalline Tate tensors $(u_{\alpha,\bar{x}})_\alpha$ on $\mathbb{D}(\mathscr{G}_{\bar{x},\mathrm{perf}})$, where $\mathscr{G}_{\bar{x},\mathrm{perf}}$ is the pullback of $\mathscr{G}_{\bar{x}}$ to $\bar{R}_G^{\mathrm{perf}}$.

Theorem 2.11. *The central leaves on $\mathscr{S}_K(G,X) \otimes \bar{\mathbb{F}}_p$ are open and closed in the naive central leaves.*

Proof: We consider the p-divisible group $\mathscr{G}_{\bar{x},\mathrm{perf}}$ over $\widehat{\mathcal{O}}^{\mathrm{perf}}_{\mathscr{S}_K(G,X)\otimes_{\mathcal{O}_{\mathbf{E},(v)}}\bar{\mathbb{F}}_p,\bar{x}}$ together with its crystalline Tate tensors. Note that perfection makes no difference for topological considerations.

On the naive central leaves on $\mathrm{Spec}(\widehat{\mathcal{O}}_{\mathscr{S}_K(G,X)\otimes_{\mathcal{O}_{\mathbf{E},(v)}}\bar{\mathbb{F}}_p,\bar{x}}) = \mathrm{Spec}(\widehat{\mathcal{O}}^{\mathrm{perf}}_{\mathscr{S}_K(G,X)\otimes_{\mathcal{O}_{\mathbf{E},(v)}}\bar{\mathbb{F}}_p,\bar{x}})$ the p-divisible group is geometrically fiberwise constant. By a lemma of Hamacher [Ham17, 2.12], the tensors then are geometrically fiberwise constant as well.

Using the fact that $\widehat{\mathcal{O}}_{\mathscr{S}_K(G,X)\otimes_{\mathcal{O}_{\mathbf{E},(v)}}\bar{\mathbb{F}}_p,\bar{x}}$ is noetherian, we obtain that the central leaf on $\widehat{\mathcal{O}}_{\mathscr{S}_K(G,X)\otimes_{\mathcal{O}_{\mathbf{E},(v)}}\bar{\mathbb{F}}_p,\bar{x}}$ is closed and open in the corresponding naive central leaf.

$\mathrm{Spec}\,\mathcal{O}_{\mathscr{S}_K(G,X)\otimes_{\mathcal{O}_{\mathbf{E},(v)}}\bar{\mathbb{F}}_p,\bar{x}}$ has the quotient topology with respect to the fpqc covering $\mathrm{Spec}\,\widehat{\mathcal{O}}_{\mathscr{S}_K(G,X)\otimes_{\mathcal{O}_{\mathbf{E},(v)}}\bar{\mathbb{F}}_p,\bar{x}} \to \mathrm{Spec}\,\mathcal{O}_{\mathscr{S}_K(G,X)\otimes_{\mathcal{O}_{\mathbf{E},(v)}}\bar{\mathbb{F}}_p,\bar{x}}$, hence we also get the same result after leaving away completion.

By Lemma 2.10 we conclude that we also get the same result for the central leaves and naive central leaves on $\mathscr{S}_K(G,X) \otimes \bar{\mathbb{F}}_p$. □

Corollary 2.12. *The central leaves on $\mathscr{S}_K(G,X) \otimes \bar{\mathbb{F}}_p$ are locally closed.*

Proof: Because Oort [Oor04] has proven that the naive central leaves on $\mathscr{S}_K(G,X) \otimes \bar{\mathbb{F}}_p$ are locally closed, this is an immediate consequence of the preceding theorem. □

Corollary 2.13. *In fact, Oort has shown that the naive central leaf is closed in the naive Newton stratum. So our argument even goes to show that the central leaves are closed in their respective Newton strata (naive and non-naive).*

2.3 Quasi-isogenies of p-divisible groups

Let $b \in G(\breve{\mathbb{Q}}_p) = \mathcal{G}(\breve{\mathbb{Q}}_p) \subseteq \mathrm{GL}(\Lambda^{\S})(\breve{\mathbb{Q}}_p)$ and denote by $b^{\S}$ the image in $\mathrm{GL}(V^{\S})(\breve{\mathbb{Q}}_p)$.

Let J_b denote the $\mathbb{Q}_p$-reductive group given on R-valued points, $R \in (\mathbb{Q}_p\text{-alg})$, by

$$J_b(R) := \left\{ g \in G(R \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p) = \mathrm{Res}_{\breve{\mathbb{Q}}_p/\mathbb{Q}_p}(G)(R) \;\middle|\; gb\sigma(g)^{-1} = b \right\}$$

(cf. [RZ96, Prop. 1.12] and [Kim19, Prop. 2.2.6]).

$J_b(\mathbb{Q}_p)$ then naturally attains the structure of a locally profinite group. We make it into a formal group scheme $\underline{J_b(\mathbb{Q}_p)}$ over $\mathrm{Spf}\,\breve{\mathbb{Z}}_p$; with $\underline{J_b(\mathbb{Q}_p)}(U) = \mathrm{Map}_{\mathrm{cont}}(U, J_b(\mathbb{Q}_p))$ for formal test schemes $U \to \mathrm{Spf}\,\breve{\mathbb{Z}}_p$.[1]

Let $\mathbb{X}_b = \mathbb{X}_{b,K} = \mathbb{X}_{b^\S}$ be a polarized p-divisible group over $\bar{\mathbb{F}}_p$ with a distinguished isomorphism between its Dieudonné (symplectic) module and $(\Lambda^\S)_{\breve{\mathbb{Z}}_p}$ under which the Frobenius is identified with b. The existence of such a p-divisible group depends on b, but we *assume* that $\mathbb{X}_b$ exists from now on (see also 2.35 (2) below). By Dieudonné theory we obtain a bijection

$$\begin{aligned} J_{b^\S}(\mathbb{Q}_p) &:= \left\{ g \in \mathrm{GL}(\Lambda^\S \otimes \breve{\mathbb{Q}}_p) \;\middle|\; gb^\S\sigma(g)^{-1} = b^\S \right\} \\ &\cong \{\text{(self-)quasi-isogenies of } \mathbb{X}_b \text{ (without polarization)}\} \end{aligned}$$

(and similar with polarization (replace GL by (G)Sp)), and $J_b(\mathbb{Q}_p) \subseteq J_{b^\S}(\mathbb{Q}_p)$ under this bijection corresponds to the *tensor-preserving* quasi-isogenies of $\mathbb{X}_b$.

We need to understand more than this; at least we will need to understand the tensor-preserving quasi-isogenies of $(\mathbb{X}_b)_\Omega$ for all algebraically closed fields of characteristic p. Using internal hom p-divisible groups first defined by Chai and Oort, Caraiani and Scholze [CS17, Prop. 4.2.11] worked out the structure of the quasi-isogeny group (albeit in a more special setting than ours; Kim [Kim19] generalized it to our setting).

First of all, following [Kim19, section 3.2], we denote by $\mathrm{Qisg}(\mathbb{X}_b)$ the formal group scheme over $\mathrm{Spf}\,\breve{\mathbb{Z}}_p$ given by

$$\mathrm{Nilp}_{\breve{\mathbb{Z}}_p} \to \mathrm{Grp}, \quad R \mapsto \{\text{quasi-isogenies of } (\mathbb{X}_b)_{R/p}\},$$

where $\mathrm{Nilp}_{\breve{\mathbb{Z}}_p}$ is the category of $\breve{\mathbb{Z}}_p$-algebras on which p is nilpotent (anti-equivalent to the category of affine schemes over $\mathrm{Spf}\,\breve{\mathbb{Z}}_p$[2]).

Definition 2.14. Let R be a topological ring.

[1] Locally: If $G = \varprojlim_n G_n$ is a profinite set (or even profinite group), then $\underline{G} := \varprojlim_n \underline{G_n}$ with $\underline{G_n}$ constant formal (group) scheme.

[2] That is, affine schemes together with a natural transformation of functors $\mathrm{Ring} \to \mathrm{Set}$ from it to $\mathrm{Spf}\,\breve{\mathbb{Z}}_p$.

(1) R is called *adic*, if there is an ideal I (called *ideal of definition*) such that $\{I^n\}_{n\in\mathbb{N}}$ is a basis of neighborhoods (or, equivalently, basis of *open* neighborhoods) of 0.

(2) R is called *f-adic* (or *Huber ring*), if there exists an open subring $A_0 \subseteq A$ that is adic with finitely generated ideal of definition. Such a ring is called a *ring of definition*.

Definition 2.15. Let R be a ring of characteristic p.

(1) R is *semiperfect*, if the Frobenius endomorphism $\Phi\colon R \to R$ is surjective.

(2) R is *f-semiperfect*, if it is semiperfect and the perfection $R^\flat := \varprojlim_\Phi R$ (with the inverse limit topology) is f-adic.

There is a functorial construction which assigns to every semiperfect ring R its universal p-adically complete pd-thickening $A_{\mathrm{cris}}(R)$ [SW13, Prop. 4.1.3]. Set $B^+_{\mathrm{cris}}(R) := A_{\mathrm{cris}}(R)[\frac{1}{p}]$. We have

$$\mathbb{D}((\mathbb{X}_b)_R)(A_{\mathrm{cris}}(R)) \cong \mathbb{D}(\mathbb{X}_b)(\breve{\mathbb{Z}}_p) \otimes_{\breve{\mathbb{Z}}_p} A_{\mathrm{cris}}(R) \cong \Lambda^\S \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}(R)$$

by construction.

Assumption 2.16. We will assume from now on that $[b] \in B(G)$ is *neutral acceptable* in the sense of [RV14, Def. 2.3], i.e., $[b] \in B(G, \{\mu\})$. (Of course this is a completely harmless assumption with regard to the setting of Shimura varieties.)

There exists an internal tensor-preserving quasi-endomorphism p-divisible group:

Lemma 2.17. [Kim19, Lemma 3.1.3] *There is a p-divisible group $\mathcal{H}_b^G$ such that for any f-semiperfect $\bar{\mathbb{F}}_p$-algebra R there is a natural $\mathbb{Q}_p$-linear isomorphism*

$$\tilde{\mathcal{H}}_b^G(R) \cong \mathrm{End}_{(s_\alpha)}((\mathbb{X}_b)_R)[\tfrac{1}{p}],$$

where $\tilde{\mathcal{H}}_b^G$ is the universal cover of $\mathcal{H}_b^G$, and where on the right hand side we have the set of $\gamma \in \mathrm{End}((\mathbb{X}_b)_R)[\frac{1}{p}]$ such that the endomorphism of $\Lambda^\S \otimes_{\mathbb{Z}_p} B^+_{\mathrm{cris}}(R)$ induced by γ preserves the tensors $s_\alpha \otimes 1$.

What we really need is an internal tensor-preserving quasi-isogeny p-divisible group. Still following [Kim19], to this end we consider the group sheaf $\mathrm{Qisg}(\mathbb{X}_b)$ on $\mathrm{Nilp}_{\breve{\mathbb{Z}}_p}$ given by $R \mapsto \{\text{self-quasi-isogenies of } (\mathbb{X}_b)_R\}$. This can be realized as a closed formal subscheme of $\tilde{\mathcal{H}}_b^2$ (where $\mathcal{H}_b := \mathcal{H}_{b^\S} = \mathcal{H}_{b^\S}^{\mathrm{GL}(V^\S)}$ (no tensors)).

The closed formal subscheme $\mathrm{Qisg}_G(\mathbb{X}_b) \subseteq \mathrm{Qisg}(\mathbb{X}_b)$ over $\mathrm{Spf}\, \breve{\mathbb{Z}}_p$ then is defined by

$$\mathrm{Qisg}_G(\mathbb{X}_b) = \mathrm{Qisg}(\mathbb{X}_b) \times_{\tilde{\mathcal{H}}_b^2} (\tilde{\mathcal{H}}_b^G)^2.$$

Proposition 2.18. [Kim19, Prop 3.2.4] *We have a natural map $\underline{J_b(\mathbb{Q}_p)} \to \mathrm{Qisg}_G(\mathbb{X}_b)$, which has a natural retraction*

$$\mathrm{Qisg}_G(\mathbb{X}_b) \to \underline{J_b(\mathbb{Q}_p)}$$

with all fibers isomorphic to $\mathrm{Spf}\, \breve{\mathbb{Z}}_p[[x_1^{p^{-\infty}}, \dots, x_d^{p^{-\infty}}]]$ as formal schemes, where $d = \langle 2\rho, \nu_{[b]} \rangle$ with 2ρ the sum of all the positive roots of $G_{\bar{\mathbb{Q}}_p}$ and $\nu_{[b]}$ the dominant[3] *representative of the $G(\breve{\mathbb{Q}}_p)$-conjugacy class of $\nu_b \in X_*(G_{\breve{\mathbb{Q}}_p})_{\mathbb{Q}}$.*[4]

Corollary 2.19. *Let $R \in \mathrm{Nilp}_{\breve{\mathbb{Z}}_p}$ be such that there is no non-trivial continuous homomorphism of $\breve{\mathbb{Z}}_p$-algebras $\breve{\mathbb{Z}}_p[[x^{p^{-\infty}}]] \to R$. Put differently, there is no $r \in R$ having a compatible system of p-power roots such that all power series with $\breve{\mathbb{Z}}_p$-coefficients in r and its p-power roots converge.*

Then

$$\mathrm{Qisg}_G(\mathbb{X}_b)(R) \cong \underline{J_b(\mathbb{Q}_p)}(R).$$

Proof: With d as in the proposition, the assumption implies that there is no non-trivial continuous homomorphism of $\breve{\mathbb{Z}}_p$-algebras $\breve{\mathbb{Z}}_p[[x_1^{p^{-\infty}}, \dots, x_d^{p^{-\infty}}]] \to R$.

Define $\mathrm{Qisg}_G^\circ(\mathbb{X}_b) := \ker(\mathrm{Qisg}_G(\mathbb{X}_b) \to \underline{J_b(\mathbb{Q}_p)}) \cong \mathrm{Spf}\, \breve{\mathbb{Z}}_p[[x_1^{p^{-\infty}}, \dots, x_d^{p^{-\infty}}]]$ and consider the split exact sequence

$$0 \to \mathrm{Qisg}_G^\circ(\mathbb{X}_b) \to \mathrm{Qisg}_G(\mathbb{X}_b) \to \underline{J_b(\mathbb{Q}_p)} \to 0. \qquad \square$$

Example 2.20. Say $R \in \mathrm{Nilp}_{\breve{\mathbb{Z}}_p}$ has characteristic p, i.e., R is a $\bar{\mathbb{F}}_p$-algebra and the p-adic topology is the discrete topology. Let r be an element of R having a compatible system of p-power roots such that all power series with $\breve{\mathbb{Z}}_p$-coefficients in r and its p-power roots converge. Then r must be nilpotent. Therefore, if $R \in \mathrm{Nilp}_{\breve{\mathbb{Z}}_p}$ has characteristic p and is reduced (e.g., is a field), then the conditions of Corollary 2.19 are satisfied.

In particular we deduce:

Example 2.21. Let $L/\bar{\mathbb{F}}_p$ be a field extension. Then

$$\mathrm{Qisg}_G(\mathbb{X}_b)(L) \cong J_b(\mathbb{Q}_p).$$

[3] This requires distinguishing a Weyl chamber as the positive one; this arises from the choice of a convenient maximal torus in [Kim19, just before Rmk. 2.1.4]. See also Notation 3.2.

[4] So $b \mapsto \nu_{[b]}$ is the *Newton map* in the parlance of [Kot97, p. 256].

2.4 Almost product structure

Notation 2.22. We shorten notation by writing

$$\mathscr{S} := \mathscr{S}_K(G,X) \otimes_{\mathcal{O}_{\mathbf{E},(v)}} \mathcal{O}_{\breve{E}} \quad \text{and} \quad \overline{\mathscr{S}} := \mathscr{S}_K(G,X) \otimes_{\mathcal{O}_{\mathbf{E},(v)}} \bar{\mathbb{F}}_p,$$

as well as

$$\mathscr{S}^{\S} := \mathscr{S}_J(\mathrm{GSp}(V^{\S}), S^{\pm,\S}) \otimes_{\mathbb{Z}_{(p)}} \breve{\mathbb{Z}}_p \quad \text{and} \quad \overline{\mathscr{S}}^{\S} := \mathscr{S}_J(\mathrm{GSp}(V^{\S}), S^{\pm,\S}) \otimes_{\mathbb{Z}_{(p)}} \bar{\mathbb{F}}_p.$$

Moreover, we denote the central leaf associated with b by $\overline{C}^b := \overline{C}^b_K := \Upsilon_K^{-1}([\![b]\!]) \subseteq \overline{\mathscr{S}}$, and by $\overline{C}^{b^{\S}} \subseteq \overline{\mathscr{S}}^{\S}$ the corresponding central leaf in $\overline{\mathscr{S}}^{\S}$.

Assumption 2.23. From now on, we assume that Axiom A from [HK17] holds.

Remark 2.24. To explain Axiom A: Consider the *affine Deligne-Lusztig variety*

$$X_\mu(b)_K(\bar{\mathbb{F}}_p) := \left\{ g\breve{K}_p \in G(\breve{\mathbb{Q}}_p)/\breve{K}_p \;\middle|\; g^{-1}b\sigma(g) \in \breve{K}_p \mathrm{Adm}(\{\mu\})\breve{K}_p \right\}$$

(we review the definition of the *admissible subset* $\mathrm{Adm}(\{\mu\})$ in section 3.1.1).

Choose a point $\bar{x} \in \overline{\mathscr{S}}(\bar{\mathbb{F}}_p)$ with a quasi-isogeny $j\colon \mathbb{X}_b \to \mathscr{A}_{\bar{x}}[p^\infty]$ compatible with the extra structure.

We have a map $i^{\S}_{(\bar{x},j)}\colon X_{\mu^{\S}}(b^{\S})(\bar{\mathbb{F}}_p) \to \overline{\mathscr{S}}^{\S}(\bar{\mathbb{F}}_p)$ given as follows: For $g\breve{J}_p \in X_{\mu^{\S}}(b^{\S})(\bar{\mathbb{F}}_p)$, there is a p-divisible group $g\mathbb{X}_b$ isogenous to $\mathbb{X}_b$ with Frobenius on the Dieudonné module given by $g^{-1}b\sigma(g)$. The quasi-isogeny $g\mathbb{X}_b \to \mathbb{X}_b \to \mathscr{A}_{\bar{x}}[p^\infty]$ lifts to a quasi-isogeny of abelian schemes[5]. We also get a polarization and a level structure away from p on the new abelian scheme, which is the image of $g\breve{J}_b$ under $i^{\S}_{(\bar{x},j)}$.

The content of Axiom A (= Assumption 2.23) now is that

$$X_{\sigma(\mu)}(b)_K(\bar{\mathbb{F}}_p) \hookrightarrow X_{\mu^{\S}}(b^{\S})(\bar{\mathbb{F}}_p) \xrightarrow{i^{\S}_{(\bar{x},j)}} \overline{\mathscr{S}}^{\S}(\bar{\mathbb{F}}_p)$$

is to factor through $\overline{\mathscr{S}}^{-}(\bar{\mathbb{F}}_p)$ such that there is a unique lift $i\colon X_{\sigma(\mu)}(b)_K(\bar{\mathbb{F}}_p) \to \overline{\mathscr{S}}(\bar{\mathbb{F}}_p)$ with $s_{\alpha,0,i(g\breve{K}_p)} = s_{\alpha,0,\bar{x}}$ for all $g\breve{K}_p \in X_{\sigma(\mu)}(b)_K(\bar{\mathbb{F}}_p)$.

Remark 2.25. Assumption 2.23 holds in the hyperspecial case [Kis17, Thm. 1.4.4], and it holds if G is residually split [Zho18, Prop. 6.4].

[5]Idea: A lift of a quasi-isogeny $y\colon A[p^\infty] \to X$, A abelian scheme, X p-divisible group, is given by $A \to A/\ker(y)$.

Remark 2.26. With Assumption 2.23 in place, the central leaf $\overline{C}^b$ of b is non-empty by [HK17, Rmk. 4.3.1].

We review some constructions from the paper [HK17]. By $\overline{\mathrm{Ig}}^{b\S}$ we denote the (special fiber of the) Igusa variety over $\overline{C}^{b\S}$, cf. [CS17, Prop. 1.12]. It parameterizes isomorphisms between the standard p-divisible group and the universal one, and it is a perfect scheme over $\bar{\mathbb{F}}_p$. Define $\overline{\mathrm{Ig}}^{b,\diamond} := \overline{\mathrm{Ig}}^{b\S} \times_{\overline{\mathscr{S}}^{\S,\mathrm{perf}}} \overline{\mathscr{S}}^{\mathrm{perf}}$. Furthermore, $\mathfrak{Ig}^{b\S}, \mathfrak{Ig}^{b,\diamond}$ are the unique flat lifts[6] of $\overline{\mathrm{Ig}}^{b\S}$ and $\overline{\mathrm{Ig}}^{b,\diamond}$ over $\mathrm{Spf}\, \mathcal{O}_{\breve{E}}$, given by the formula

$$\mathfrak{Ig}(R) := \overline{\mathrm{Ig}}(R/\varpi) \quad \text{for all } R \in \mathrm{Nilp}_{\mathcal{O}_{\breve{E}}} \tag{2.27}$$

in both cases, where ϖ is a uniformizer of $\mathcal{O}_{\breve{E}}$.

By definition, we have an isomorphism $j \colon \mathbb{X}_b \times \overline{\mathrm{Ig}}^{b,\diamond} \xrightarrow{\cong} \mathscr{A}_{\overline{C}^b}[p^\infty] \times_{\overline{C}^b} \overline{\mathrm{Ig}}^{b,\diamond}$ and $\overline{\mathrm{Ig}}^b$ is by definition the locus of geometric points of $\overline{\mathrm{Ig}}^{b,\diamond}$, where j respects the crystalline Tate tensors. This is a closed union of connected components of $\overline{\mathrm{Ig}}^{b,\diamond}$ [HK17, Def./Lemma 5.1.1].

$\mathrm{Qisg}(\mathbb{X}_b)$ acts on $\overline{\mathrm{Ig}}^{b\S}$ and $\mathfrak{Ig}^{b\S}$, $\overline{\mathrm{Ig}}^b \to \overline{\mathrm{Ig}}^{b\S}$ is a closed immersion with $\mathrm{Qisg}_G(\mathbb{X}_b)_{\bar{\mathbb{F}}_p}$-stable image, and $\mathfrak{Ig}^b \to \mathfrak{Ig}^{b\S}$ is a closed immersion with $\mathrm{Qisg}_G(\mathbb{X}_b)$-stable image [HK17, Prop. 5.1.2, Cor 5.1.3].

Let $\mathfrak{M}^{b\S} \to \mathrm{Spf}\, \breve{\mathbb{Z}}_p$ be the Rapoport-Zink space given by

$$\mathfrak{M}^{b\S}(R) = \left\{ (\mathscr{X}, \rho) \;\middle|\; \begin{array}{l} \mathscr{X}/R \text{ polarized } p\text{-divisible group and} \\ \rho \colon \mathbb{X}_b \otimes R/p \to \mathscr{X} \otimes R/p \text{ quasi-isogeny resp. polarizations} \end{array} \right\}.$$

Choose a point $\bar{x} \in \overline{\mathscr{S}}(\bar{\mathbb{F}}_p)$ with a quasi-isogeny $j \colon \mathbb{X}_b \to \mathscr{A}_{\bar{x}}[p^\infty]$ compatible with the extra structure.

$\mathfrak{M}^{b\S}$ comes with the Rapoport-Zink uniformization map $\Theta^\S \colon \mathfrak{M}^{b\S} \to \mathscr{S}^\S$ depending on this choice [RZ96, (6.3)]: Given $(\mathscr{X}, \rho)$ as above, ρ lifts to a quasi-isogeny $\tilde{\rho} \colon \mathbb{X}_b \otimes R \to \mathscr{X}$, and there is an abelian scheme $\mathscr{Y}$ with p-divisible group $\mathscr{X}$ and a unique lift $\tilde{\mathscr{A}}_{\bar{x}} \otimes R \to \mathscr{Y}$ of $\tilde{\rho}$ for a chosen lift $\tilde{\mathscr{A}}_{\bar{x}}$ of $\mathscr{A}_{\bar{x}}$ to $\breve{\mathbb{Z}}_p$. We also get a polarization and a level structure away from p on $\mathscr{Y}$, and $\mathscr{Y}$ with these extra structures is the image of $(\mathscr{X}, \rho)$ under $\Theta^\S$.

Remark 2.28. The affine Deligne-Lusztig variety is the perfection of the special fiber of the Rapoport-Zink space [Zhu17, Prop. 0.4]. Under this isomorphism, $\Theta^\S$ corresponds to $i^\S$ from Remark 2.24.

[6] Locally: Let R be a perfect $\bar{\mathbb{F}}_p$-algebra. Then $W_{\mathcal{O}_E}(R)$ (ramified Witt vectors, cf. Remark 1.17) is the unique ϖ-adically complete $\mathcal{O}_{\breve{E}}$-flat algebra lifting it (cf. [Ahs11, Prop. 1.3.3]).

Define $\mathfrak{M}^{b,\diamond} := \mathfrak{M}^{b^\S} \times_{\mathscr{S}^\S} \mathscr{S}$ and define $\Theta^\diamond \colon \mathfrak{M}^{b,\diamond} \to \mathscr{S}$ to be the base change of $\Theta^\S$.

Let $\mathscr{X}^\diamond \to \mathfrak{M}^{b,\diamond}$ be the pullback of the universal p-divisible group over $\mathfrak{M}^{b^\S}$. Then we have two families of tensors on $\mathbb{D}(\mathscr{X}^\diamond)[\frac{1}{p}]$ (or more precisely, two families of maps $\mathfrak{M}^{b,\diamond}(\bar{\mathbb{F}}_p) \ni \bar{y} \mapsto$ tensor on $\mathbb{D}(\mathscr{X}^\diamond_{\bar{y}})[\frac{1}{p}]$):

(1) $(t^\diamond_\alpha)_\alpha$ obtained from $(s_\alpha)_\alpha$ via $\mathbb{D}(\mathbb{X}_b \otimes \mathfrak{M}^{b,\diamond})[\frac{1}{p}] \cong \mathbb{D}(\mathscr{X}^\diamond)[\frac{1}{p}]$, and

(2) for every $\bar{y} \in \mathfrak{M}^{b,\diamond}(\bar{\mathbb{F}}_p)$, $\left(u^\diamond_{\alpha,\bar{y}}\right)_\alpha$ with $u^\diamond_{\alpha,\bar{y}} := (\Theta^\diamond)^* s_{\alpha,0,\Theta^\diamond(\bar{y})}$.

$\mathfrak{M}^b$ is defined to be the formal subscheme of $\mathfrak{M}^{b,\diamond}$ corresponding to the locus where these agree. Also, $\Theta \colon \mathfrak{M}^b \to \mathscr{S}$ is defined to be the restriction of $\Theta^\diamond$. Moreover, $\mathfrak{M}^b$ has a natural $\mathrm{Qisg}_G(\mathbb{X}_b)$-action.

By [CS17, § 4], there is an isomorphism

$$\mathfrak{Ig}^{b^\S} \times \mathfrak{M}^{b^\S} \xrightarrow{\sim} \mathfrak{X}^{b^\S} \tag{2.29}$$

with the *Newton-Igusa variety*

$$\mathfrak{X}^{b^\S}(R) = \left\{ (A, \lambda, \eta^p; \psi) \;\middle|\; \begin{array}{l} (A, \lambda, \eta^p) \in \mathscr{S}^\S(R), \\ \psi \colon (\mathbb{X}_b, \lambda_{\mathbb{X}_b}) \otimes R/p \to (A[p^\infty], \lambda) \otimes R/p \text{ quasi-isogeny} \end{array} \right\}.$$

Remark 2.30. Let us quickly recall how the isomorphism (2.29) works (suppressing polarizations in the notation for simplicity). Let $(\mathcal{A}, \xi) \in \mathfrak{Ig}^{b^\S}(R)$ and $(\mathscr{X}, \rho) \in \mathfrak{M}^{b^\S}(R)$ be given. Consider the composition

$$\mathscr{X} \xrightarrow{\rho^{-1}} \mathbb{X}_b \otimes R \xrightarrow{\xi} \mathcal{A}[p^\infty], \tag{2.31}$$

where we denote a lift $\mathbb{X}_b \otimes R/p \xrightarrow{\rho} \mathscr{X} \otimes R/p$ by ρ again. Lift (2.31) to a quasi-isogeny of abelian schemes $\mathcal{A}' \to \mathcal{A}$. Then $(\mathcal{A}', \rho) \in \mathfrak{X}^{b^\S}(R)$ is our image point.

Define $\mathfrak{X}^b \subseteq \mathfrak{X}^{b^\S}$ to be the image of $\mathfrak{Ig}^b \times \mathfrak{M}^b$ under the isomorphism (2.29). This comes with a natural $\mathrm{Qisg}_G(\mathbb{X}_b)$-action; cf. [HK17, section 5.2].

We have canonical maps $\pi^\S_\infty \colon \mathfrak{X}^{b^\S} \to \mathscr{S}^\S$ and $\pi_\infty \colon \mathfrak{X}^b \to \mathscr{S}$.

Remark 2.32. By [HK17, Thm. 5.2.6 (1)], $\pi^{\mathrm{perf}}_{\infty,\bar{\mathbb{F}}_p} \colon \mathfrak{X}^{b,\mathrm{perf}}_{\bar{\mathbb{F}}_p} \to \overline{\mathscr{S}}^{\mathrm{perf}}$ represents the moduli problem

$$\begin{aligned} &(\text{perfect affine } \overline{\mathscr{S}}^{\mathrm{perf}}\text{-schemes}) \to \mathrm{Set}, \\ &\qquad (\mathrm{Spec}(R) \xrightarrow{Q} \overline{\mathscr{S}}^{\mathrm{perf}}) \mapsto \left\{ \begin{array}{c} \psi \colon (\mathbb{X}_b)_R \to (\mathscr{G}_{\overline{\mathscr{S}}^{\mathrm{perf}}})_Q \text{ quasi-isogeny} \\ \text{compatible with crystalline Tate-tensors} \end{array} \right\} \end{aligned}$$

with $\mathscr{G}_{\overline{\mathscr{S}}^{\mathrm{perf}}}$ and crystalline Tate tensors on the right hand side as in Proposition 1.68.

2.5 Change of parahoric level

Now we consider the question of how the central leaves behave when the level at p is changed from K_p to a larger K_p', where K_p and K_p' are associated with points $x, x' \in \mathcal{B}(G, \mathbb{Q}_p)$ as described in Chapter 1. Define $K := K_p K^p$, as usual, and $K' := K_p' K^p$. Let $b \in G(\breve{\mathbb{Q}}_p)$.

Let $(\mathcal{L}, c)$ and $(\mathcal{L}', c')$ be the graded lattice chains that are the images of x and x', respectively, under the embedding of buildings $\iota\colon \mathcal{B}(G, \mathbb{Q}_p) \hookrightarrow \mathcal{B}(\mathrm{GSp}(V), \mathbb{Q}_p)$ described in Section 1.2. Note that the gradings c and c' play no role for our purposes.

Remarks 2.33. (1) Observe that K_p and K_p' depend only on the minimal facet $\mathfrak{f}$ and $\mathfrak{f}'$ with $x \in \mathfrak{f}$ and $x' \in \mathfrak{f}'$, respectively. The inclusion $K_p \subseteq K_p'$ means precisely that $\mathfrak{f}' \subseteq \bar{\mathfrak{f}}$. We can move x' arbitrarily close to x without altering K_p'. The minimal facets $\mathfrak{g}$ and $\mathfrak{g}'$ of $\mathcal{B}(\mathrm{GSp}(V), \mathbb{Q}_p)$ containing $\iota(x)$ and $\iota(x')$, respectively, do *not* depend only on $\mathfrak{f}, \mathfrak{f}'$. Still, for x' sufficiently close to x, we may assume that $\mathfrak{g}' \subseteq \bar{\mathfrak{g}}$ (if it were not so, we would not be able to move $\iota(x')$ arbitrarily close to $\iota(x)$ — but recall that ι is continuous), i.e., that $\mathcal{L}'$ is a thinned out version of $\mathcal{L}$. Say $\mathcal{L}$ is given by

$$\Lambda^0 \supsetneqq \Lambda^1 \supsetneqq \cdots \supsetneqq \Lambda^{r-1} \supsetneqq p\Lambda^0$$

(as in (1.13)); then $\mathcal{L}'$ is given by $(\Lambda^{i_j})_j$ for some $0 \le i_1 < i_2 < \cdots < i_s < r$.

(2) In proofs, we can often reduce to the case where "$\mathcal{L}$ and $\mathcal{L}'$ differ by one element", which of course is supposed to mean that $s = r - 2$ in this notation.

Notation 2.34. We define $N_p, N_p', J_p, J_p', \Lambda^{\S}, {\Lambda'}^{\S}, V^{\S}, {V'}^{\S}$ as in Section 1.4.

2.5.1 The change-of-parahoric morphism

As explained in Section 1.6, we obtain (for K^p sufficiently small and some compact open subgroup $N^p \subseteq \mathrm{GSp}(V)(\mathbb{A}_f^p)$) a commutative diagram

$$\begin{tikzcd}
\mathscr{S}_{K_pK^p}(G,X) \arrow[r, "\pi_{K_pK^p,K'_pK^p}"] \arrow[d, "\text{finite}"'] & \mathscr{S}_{K'_pK^p}(G,X) \arrow[d, "\text{finite}"] \\
\mathscr{S}_{N_pN^p}(\mathrm{GSp}(V),S^{\pm})_{\mathcal{O}_{\mathbf{E},(v)}} \arrow[r, "\pi_{N_pN^p,N'_pN^p}"] & \mathscr{S}_{N'_pN^p}(\mathrm{GSp}(V),S^{\pm})_{\mathcal{O}_{\mathbf{E},(v)}}
\end{tikzcd}$$

Notation 2.35. (1) The change-of-parahoric map $\pi_{K,K'} := \pi_{K_pK^p,K'_pK^p}$ restricts to a change-of-parahoric map $\Upsilon^{-1}_{K_pK^p}([\![b]\!]) \to \Upsilon^{-1}_{K'_pK^p}([\![b]\!])$ between leaves, which we denote by $\pi_{K,K'}|_{\Upsilon_K^{-1}([\![b]\!])}$ or simply by $\pi_{K,K'}$ again.

(2) We choose a base point $\bar{x}_b \in \overline{C}^b_K(\bar{\mathbb{F}}_p)$ and take its image under $\pi_{K,K'}$ as a base point $\bar{x}'_b \in \overline{C}^b_{K'}(\bar{\mathbb{F}}_p)$. By $\mathbb{X}_{b,K}$ and $\mathbb{X}_{b,K'}$, respectively, we denote the corresponding polarized p-divisible groups.

(3) Denote by $\mathscr{A}_K$ the universal polarized abelian scheme over $\overline{\mathscr{S}}_K$. By slight abuse of notation, we will also use the same notation for its pullback to $\overline{\mathscr{S}}_K^{\mathrm{perf}}$.

Remark 2.36. We have morphisms $\mathbb{X}_{b,K} \to \mathbb{X}_{b,K'}$ and, more generally, $\mathscr{A}_K \to \mathscr{A}_{K'}$ (lying over $\overline{\mathscr{S}}_K \to \overline{\mathscr{S}}_{K'}$). This follows from the "thinning out" interpretation, Remark 2.33 (1).

2.5.2 Newton-Igusa variety and change-of-parahoric

Lemma 2.37. *There are natural change-of-parahoric morphisms*

$$\mathfrak{Ig}^b_K \to \mathfrak{Ig}^b_{K'}, \quad \mathfrak{M}^b_K \to \mathfrak{M}^b_{K'}, \quad \mathfrak{X}^b_K \to \mathfrak{X}^b_{K'}.$$

lying over the change-of-parahoric morphism $\mathscr{S}_K \to \mathscr{S}_{K'}$ and compatible with the Siegel embeddings.

PROOF: Consider the universal isomorphism $j\colon \mathbb{X}_{b,K} \times \overline{\mathrm{Ig}}_K^{b,\diamond} \xrightarrow{\cong} \mathscr{A}_{K,\overline{C}_K^b}[p^\infty] \times_{\overline{C}_K^b} \overline{\mathrm{Ig}}_K^{b,\diamond} = \mathscr{A}_K[p^\infty] \times_{\overline{\mathscr{S}}_K^{\mathrm{perf}}} \overline{\mathrm{Ig}}_K^{b,\diamond}$ and the diagram

$$\begin{tikzcd}
\mathbb{X}_{b,K} \times \overline{\mathrm{Ig}}_K^{b,\diamond} \arrow[r, "\cong"] \arrow[d] & \mathscr{A}_K[p^\infty] \times_{\overline{\mathscr{S}}_K^{\mathrm{perf}}} \overline{\mathrm{Ig}}_K^{b,\diamond} \arrow[d] \\
\mathbb{X}_{b,K'} \times \overline{\mathrm{Ig}}_K^{b,\diamond} \arrow[r, dashed, "\cong"] & \mathscr{A}_{K'}[p^\infty] \times_{\overline{\mathscr{S}}_{K'}^{\mathrm{perf}}} \overline{\mathrm{Ig}}_K^{b,\diamond}
\end{tikzcd}$$

The dashed arrow exists on $\overline{\mathrm{Ig}}_K^b$, i.e., $\mathbb{X}_{b,K'} \times \overline{\mathrm{Ig}}_K^b \xrightarrow{\cong} \mathscr{A}_{K'}[p^\infty] \times_{\overline{\mathscr{S}}_{K'}^{\mathrm{perf}}} \overline{\mathrm{Ig}}_K^b$ exists. The reason is that the tensors are respected on $\overline{\mathrm{Ig}}_K^b$ which in particular forces $\mathbb{X}_{b,K} \times \overline{\mathrm{Ig}}_K^{b,\diamond} \xrightarrow{\cong} \mathscr{A}_K[p^\infty] \times_{\overline{\mathscr{S}}_K^{\mathrm{perf}}} \overline{\mathrm{Ig}}_K^{b,\diamond}$ to be "diagonal", similar to the proof of Proposition 1.26.

This isomorphism $\mathbb{X}_{b,K'} \times \overline{\mathrm{Ig}}_K^b \xrightarrow{\cong} \mathscr{A}_{K'}[p^\infty] \times_{\overline{\mathscr{S}}_{K'}^{\mathrm{perf}}} \overline{\mathrm{Ig}}_K^b$ now corresponds to a morphism $\overline{\mathrm{Ig}}_K^b \to \overline{\mathrm{Ig}}_{K'}^b$. By construction (2.27) of $\mathfrak{Ig}^b$, we also get a corresponding morphism $\mathfrak{Ig}_K^b \to \mathfrak{Ig}_{K'}^b$.

For $\mathfrak{X}$ and $\mathfrak{M}$, it is very similar. □

Lemma 2.38. *The isomorphism $\mathfrak{Ig}^b \times \mathfrak{M}^b \cong \mathfrak{X}^b$ is compatible with change-of-parahoric, where the change-of-parahoric map for $\mathfrak{Ig}^b \times \mathfrak{M}^b$ is by definition the product of those for $\mathfrak{Ig}^b$ and $\mathfrak{M}^b$, respectively.*

PROOF: All of $\mathfrak{Ig}^b, \mathfrak{M}^b, \mathfrak{X}^b$ are embedded into the corresponding Siegel versions as closed formal subschemes, compatible with change-of-parahoric. Hence we may assume that we are in the Siegel case.

In that case, the isomorphism has a very concrete description, cf. Remark 2.30, from which the lemma is clear. □

Definition 2.39. By $\overline{\mathscr{S}}_K^b := \overline{\mathscr{S}}_K^{[b]}$ we denote the Newton stratum associated with $[b] \in B(G)$, i.e., $\bar{S}_{K,[b]}$ in the notation of [HR17].

Lemma 2.40. *Let $\Omega/\mathbb{F}_p$ be an algebraically closed field.*

The action of $J_b(\mathbb{Q}_p)$ on the fibers of $\mathfrak{X}^b_K(\Omega) \to \overline{\mathscr{S}}^b_K(\Omega)$ is simply transitive and the change-of-parahoric map

$$\mathfrak{X}^b_K(\Omega) \to \mathfrak{X}^b_{K'}(\Omega)$$

is $J_b(\mathbb{Q}_p)$-equivariant.

PROOF: The moduli description of Remark 2.32 gives us that the action is simply transitive upon noting that

$$\mathrm{Qisg}_G(\mathbb{X}_{b,K})(\Omega) \overset{2.21}{\cong} J_b(\mathbb{Q}_p) \overset{2.21}{\cong} \mathrm{Qisg}_G(\mathbb{X}_{b,K'})(\Omega).$$

Equivariance follows from the description of the action of $J_b(\mathbb{Q}_p)$ and of the change-of-parahoric map in terms of lattice chains in Dieudonné theory: Given a point $Q\colon \operatorname{Spec}\Omega \to \overline{\mathscr{S}}^b_K$, we consider the map between the fiber of Q under $\mathfrak{X}^b_K(\Omega) \to \overline{\mathscr{S}}^b_K(\Omega)$ and the fiber of the image of Q under $\mathfrak{X}^b_{K'}(\Omega) \to \overline{\mathscr{S}}^b_{K'}(\Omega)$. We identify the lattice chain associated with Q with a standard lattice chain such that the Frobenius is identified with a $b' \in G(\breve{\mathbb{Q}}_p)$, $b' \equiv b \mod \breve{K}_{p,\sigma}$. Elements of $J_b(\mathbb{Q}_p)$ then act naturally on the common rational Dieudonné module. The action on the fiber of Q is given by altering the quasi-isogenies appearing in the description of that set by the quasi-isogeny obtained this way. Passing from K to K', i.e., from Q to the image of Q, means leaving out parts of the lattice chain and enlarging $\breve{K}_p$ (so one still has the same b'). □

Proposition 2.41. *The change-of-parahoric map*

$$\mathfrak{X}^b_{K,\bar{\mathbb{F}}_p} \to \mathfrak{X}^b_{K',\bar{\mathbb{F}}_p}$$

is surjective.

PROOF: We check that

$$\mathfrak{X}^b_K(\Omega) \to \mathfrak{X}^b_{K'}(\Omega)$$

is surjective for every algebraically closed field $\Omega/\mathbb{F}_p$.

Considering the diagram

$$\begin{tikzcd}
\mathfrak{X}^b_K(\Omega) \arrow[r] \arrow[d] & \mathfrak{X}^b_{K'}(\Omega) \arrow[d] \\
\overline{\mathscr{S}}^b_K(\Omega) \arrow[r, two heads] & \overline{\mathscr{S}}^b_{K'}(\Omega)
\end{tikzcd}$$

this follows from the preceding lemma and the fact[7] that the change-of-parahoric map between Newton strata is surjective. □

Corollary 2.42. *The map* $\overline{\mathrm{Ig}}^b_K \to \overline{\mathrm{Ig}}^b_{K'}$ *is an isomorphism.*

Proof: Lemma 2.38 and Proposition 2.41 imply that it is surjective.

Also, it is the restriction to closed subschemes of the isomorphism $\overline{\mathrm{Ig}}^{b\S}_K \to \overline{\mathrm{Ig}}^{b\S}_{K'}$. Since all involved schemes are reduced, the corollary follows. □

Corollary 2.43. *The change-of-parahoric morphism between central leaves is surjective.*

Proof: We have a diagram

$$\begin{tikzcd}
\overline{\mathrm{Ig}}^b_K \arrow[r, "\cong"] \arrow[d, two heads] & \overline{\mathrm{Ig}}^b_{K'} \arrow[d, two heads] \\
\Upsilon_K^{-1}(b) \arrow[r] & \Upsilon_{K'}^{-1}(b)
\end{tikzcd}$$

where we already know that all maps but the lower horizontal one are surjective. □

Corollary 2.44. *The separable rank of* $\Upsilon_K^{-1}(b) \to \Upsilon_{K'}^{-1}(b)$*, i.e., the number of geometric points of the fibers, is finite and constant.*

Proof: We have

$$\Upsilon_K^{-1}(b) \cong \overline{\mathrm{Ig}}^b_K/\operatorname{Aut}(\mathbb{X}_{b,K}) \cong \overline{\mathrm{Ig}}^b_{K'}/\operatorname{Aut}(\mathbb{X}_{b,K}) \quad \text{and} \quad \Upsilon_{K'}^{-1}(b) \cong \overline{\mathrm{Ig}}^b_{K'}/\operatorname{Aut}(\mathbb{X}_{b,K'}),$$

so that all fibers are isomorphic to $\operatorname{Aut}(\mathbb{X}_{b,K'})/\operatorname{Aut}(\mathbb{X}_{b,K})$. □

Corollary 2.45. *The change-of-parahoric map between central leaves is finite.*

[7] This follows simply from the surjectiveness of $\overline{\mathscr{S}}_K \to \overline{\mathscr{S}}_{K'}$ and the commutativity of the diagram $\begin{tikzcd}[column sep=small] \overline{\mathscr{S}}_K \arrow[rr] \arrow[dr] & & \overline{\mathscr{S}}_{K'} \arrow[dl] \\ & B(G) & \end{tikzcd}$, non-horizontal maps being the Newton maps.

PROOF: We have just seen it to be quasi-finite.

Corollary 2.13, combined with the properness of the change-of-parahoric itself and therefore of the change-of-parahoric map restricted to Newton strata (similar argument as in footnote 7), implies the properness of the change-of-parahoric map between central leaves. □

Remark 2.46. By [Kim19, Cor. 5.3.1], leaves are equidimensional smooth and the dimension of a leaf depends only on the Newton stratum it is in; in particular, if we consider a change-of-parahoric map between leaves,

$$\pi_{K,K'} \colon \Upsilon_K^{-1}(y) \to \Upsilon_{K'}^{-1}(y'),$$

then $\dim \Upsilon_K^{-1}(y) = \dim \Upsilon_{K'}^{-1}(y')$.

Corollary 2.47. *The change-of-parahoric map between central leaves is finite locally free.*

PROOF: Combine Corollary 2.45 with the preceding remark and [GW10, Cor. 14.128][8]. □

Corollary 2.48. *The change-of-parahoric morphism between central leaves is the composition of a flat universal homeomorphism of finite type and a finite étale morphism.*

PROOF: This follows from what has been established about the morphism by using [Mes72, Lemma 4.8]. □

[8]Note that "$y \in Y$" may be replaced by "$y \in f(X)$" in the statement of [GW10, Cor. 14.128].

3 EKOR strata and zips in the case of parahoric reduction

Notation 3.1. We still fix a Shimura datum (G, X) of Hodge type, a parahoric subgroup $K_p \subseteq G(\mathbb{Q}_p)$ (associated with a Bruhat-Tits group scheme $\mathcal{G} = \mathcal{G}_K = \mathcal{G}_{K_p} \to \operatorname{Spec} \mathbb{Z}_p$ associated with a facet $\mathfrak{f}$) and a sufficiently small open compact subgroup $K^p \subseteq G(\mathbb{A}_f^p)$. Define $\overline{\mathcal{G}}_K := \mathcal{G}_K \otimes_{\mathbb{Z}_p} \kappa$.

We also keep up our standard assumptions 1.18.

We now want to discuss the EKOR stratification on the special fiber of the integral model with parahoric level structure. The EKOR stratification interpolates between the Ekedahl-Oort (EO) and the Kottwitz-Rapoport (KR) stratification (see Remark 3.22 below for a precise formulation). We begin by explaining the basics about these stratifications and the combinatorics involved in the first section of this chapter.

3.1 The Ekedahl-Oort, Kottwitz-Rapoport and EKOR stratifications

3.1.1 Iwahori-Weyl group and the admissible subset

Notation 3.2. (1) We fix an Iwahori subgroup $I_p \subseteq K_p$, i.e., I_p is the group of $\mathbb{Z}_p$-points of the parahoric group scheme $\mathcal{I}$ associated with an alcove $\mathfrak{a}$ (facet of maximal dimension) such that $\mathfrak{f} \subseteq \overline{\mathfrak{a}}$. As usual, we also define $\breve{I} := \mathcal{I}(\breve{\mathbb{Z}}_p) \subseteq \breve{K}$.

(2) Let $T \subseteq G$ be a maximal torus such that $T_{\breve{\mathbb{Q}}_p}$ is contained in a Borel subgroup of $G_{\breve{\mathbb{Q}}_p}$[1] and let S be the maximal $\breve{\mathbb{Q}}_p$-split torus contained in $T_{\breve{\mathbb{Q}}_p}$. We can and do choose T such that the alcove $\mathfrak{a}$ is contained in the apartment associated with S. By N we denote the normalizer of T.

[1] Note that by Steinberg's theorem, $G_{\breve{\mathbb{Q}}_p}$ is quasi-split. [Ser97, Chap. III, § 2]

(3) Let (V, R) be the relative root system of $(G_{\breve{\mathbb{Q}}_p}, T_{\breve{\mathbb{Q}}_p})$, i.e., V is the $\mathbb{R}$-vector space $X^*_{\breve{\mathbb{Q}}_p}(T_{\breve{\mathbb{Q}}_p}) \otimes_{\mathbb{Z}} \mathbb{R}$ and $R \subseteq X^*_{\breve{\mathbb{Q}}_p}(T_{\breve{\mathbb{Q}}_p})$ is (as usual) such that we have a decomposition

$$\mathfrak{g} := \mathrm{Lie}(G_{\bar{\mathbb{Q}}_p}) = \mathrm{Lie}(T_{\bar{\mathbb{Q}}_p}) \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha.$$

Contrary to the absolute situation, $\dim \mathfrak{g}_\alpha$ may be greater than 1.

Definition 3.3. (1) The *(finite relative) Weyl group* of G (over $\breve{\mathbb{Q}}_p$) is $W := N(\breve{\mathbb{Q}}_p)/T(\breve{\mathbb{Q}}_p)$. It is the Weyl group of the root system (V, R), i.e., the group generated by the orthogonal reflections through the hyperplanes defined by the elements of R.

(2) As described in [Lan00, 1.2.3], one defines a set of affine roots $R_{\mathrm{aff}} \supseteq R$ on V using the valuation on $\breve{\mathbb{Q}}_p$. By $W_a \subseteq \mathrm{Aff}(V^*) = \mathrm{GL}(V^*) \ltimes V^*$ we denote the *affine Weyl group* of the affine root system (V, R_{aff}), i.e., the group generated by the orthogonal reflections through the affine hyperplanes defined by the elements of R_{aff}.

(3) $\widetilde{W} := N(\breve{\mathbb{Q}}_p)/(T(\breve{\mathbb{Q}}_p) \cap \breve{I})$ is the *Iwahori-Weyl group*.

(4) $W_K := (N(\breve{\mathbb{Q}}_p) \cap \breve{K})/(T(\breve{\mathbb{Q}}_p) \cap \breve{I}) \subseteq \widetilde{W}$. (Recall that $\breve{K} = \mathcal{G}(\breve{\mathbb{Z}}_p)$.)

Remarks 3.4. (1) We have $W \subseteq W_a$. With the systems of generators indicated above, W and W_a become (affine) Coxeter groups; in particular we can talk about reduced words and have length functions, cf. [BB05].

(2) W_I is the trivial group.

Proposition 3.5. [HR08, Prop. 8] *The Bruhat-Tits decomposition*

$$G(\breve{\mathbb{Q}}_p) = \bigcup_{w \in \widetilde{W}} \breve{K} w \breve{K}$$

identifies

$$\breve{K} \backslash G(\breve{\mathbb{Q}}_p)/\breve{K} \cong W_K \backslash \widetilde{W} / W_K.$$

Proposition 3.6. [HR08, Prop. 13] *Let $\breve{K}$ be the maximal parahoric subgroup of $G(\breve{\mathbb{Q}}_p)$ associated with a special vertex in the apartment corresponding to S. Then $W_K \to W$ is an isomorphism and $\widetilde{W} \cong W \ltimes X_*(T)_{\mathrm{Gal}(\bar{\mathbb{Q}}_p/\breve{\mathbb{Q}}_p)}$.*[2]

[2] Notation: Let Γ be a group and M a $\mathbb{Z}[\Gamma]$-module. Then $M_\Gamma := \mathbb{Z} \otimes_{\mathbb{Z}[\Gamma]} M = M/\langle \gamma m - m \mid \gamma \in \Gamma,\ m \in M \rangle$ is the module of Γ-coinvariants of M.

Notation 3.7. We denote the map $X_*(T)_{\mathrm{Gal}(\bar{\mathbb{Q}}_p/\breve{\mathbb{Q}}_p)} \to \widetilde{W}$ of the proposition by $\nu \mapsto t_\nu$.

Proposition 3.8. [HR08, Lemma 14] *Let $\Omega \subseteq \widetilde{W}$ be the subgroup consisting of those elements that preserve the base alcove $\mathfrak{a}$.*

There is an exact sequence

$$1 \to W_a \to \widetilde{W} \to \Omega \to 1,$$

with a canonical right splitting (namely the inclusion $\Omega \hookrightarrow \widetilde{W}$), i.e., $\widetilde{W} \cong W_a \rtimes \Omega$.

Definition 3.9. The semidirect product decomposition of the preceding proposition means that $\widetilde{W}$ is a "quasi-Coxeter" group. In practical terms, this means:

(1) We define a length function ℓ on $\widetilde{W}$ as follows: $\ell(w_a, \omega) := \ell(w_a)$ for all $w_a \in W_a$ and $\omega \in \Omega$, where on the right hand side we use the length function of the affine Coxeter group W_a.

Note that $\Omega = \ell^{-1}(0)$.

(2) Likewise, we extend the Bruhat partial order from W_a to $\widetilde{W}$ by defining

$$(w_{a,1}, \omega_1) \leq (w_{a,2}, \omega_2) \;:\Longleftrightarrow\; w_{a,1} \leq w_{a,2} \text{ and } \omega_1 = \omega_2.$$

Note that $w_1 \leq w_2$ ($w_1, w_2 \in \widetilde{W}$) implies $\ell(w_1) \leq \ell(w_2)$.

Definition 3.10. (1) Let $\{\mu\}$ be a W_{abs}-conjugacy class of geometric cocharacters of T (cf. Remark 1.4), $W_{\mathrm{abs}} := N(\bar{\mathbb{Q}}_p)/T(\bar{\mathbb{Q}}_p)$ being the absolute Weyl group. Let $\bar{\mu} \in X_*(T)_{\mathrm{Gal}(\bar{\mathbb{Q}}_p/\breve{\mathbb{Q}}_p)}$ be the image of a cocharacter in $\{\mu\}$ whose image in $X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ is contained in the closed Weyl chamber corresponding to some Borel subgroup of G containing T and defined over $\breve{\mathbb{Q}}_p$.

(2) $\mathrm{Adm}(\mu) := \mathrm{Adm}(\{\mu\}) := \{w \in \widetilde{W} \mid w \leq q t_{\bar{\mu}} q^{-1} = t_{q\bar{\mu}} \text{ for some } q \in W\}$ is the *$\{\mu\}$-admissible subset* of $\widetilde{W}$.

(3) $\mathrm{Adm}(\{\mu\})^K := W_K \mathrm{Adm}(\{\mu\}) W_K \subseteq \widetilde{W}$.

(4) $\mathrm{Adm}(\{\mu\})_K := \mathrm{KR}(K, \{\mu\}) := W_K \backslash \mathrm{Adm}(\{\mu\})^K / W_K \subseteq W_K \backslash \widetilde{W} / W_K$.

(5) Define ${}^K\widetilde{W} \subseteq \widetilde{W}$ to be the set of representatives of minimal length for the quotient $W_K \backslash \widetilde{W}$.

(6) ${}^K\mathrm{Adm}(\{\mu\}) := \mathrm{EKOR}(K, \{\mu\}) := \mathrm{Adm}(\{\mu\})^K \cap {}^K\widetilde{W} \subseteq {}^K\widetilde{W}$.

Lemma 3.11. (See [SYZ19, Thm. 1.2.2].) ${}^K\mathrm{Adm}(\{\mu\}) = \mathrm{Adm}(\{\mu\}) \cap {}^K\widetilde{W}$.

3.1.2 Kottwitz-Rapoport stratification

Recall from Section 1.9.2 that we have an integral model and a local model diagram

$$\mathscr{S}_K \leftarrow \tilde{\mathscr{S}}_K \rightarrow M_K^{\mathrm{loc}}$$

or, equivalently, a (smooth) morphism of stacks $\mathscr{S}_K \to [\mathcal{G}_K \backslash M_K^{\mathrm{loc}}]$ (over $\mathcal{O}_E$).

As explained in Section 1.9.3, by the construction in [PZ13], the special fiber $M_K^{\mathrm{loc}} \otimes \kappa$ of M_K^{loc} is a closed subscheme of the affine flag variety $\mathrm{Gr}_{\underline{\mathcal{G}}_K \otimes_{\mathbb{Z}_p} \kappa} = \mathcal{F}l_{\underline{\mathcal{G}}_K \otimes \kappa[[t]]}$, which is the ind-projective ind-scheme over κ given as the fpqc sheafification (which exists in this case!) of the presheaf $R \mapsto \underline{\mathcal{G}}_K(R((t)))/\underline{\mathcal{G}}_K(R[[t]])$.

Definition 3.12. Define $L^+(\underline{\mathcal{G}}_K \otimes \kappa[[t]])$ to be the κ-functor sending a κ-algebra R to $\underline{\mathcal{G}}_K(R[[t]])$.

We let $L^+(\underline{\mathcal{G}}_K \otimes \kappa[[t]])$ act on $\mathrm{Gr}_{\underline{\mathcal{G}}_K \otimes_{\mathbb{Z}_p} \kappa}$ from the left and call this action a (within this subsection). The orbits of this action on $\mathrm{Gr}_{\underline{\mathcal{G}}_K \otimes_{\mathbb{Z}_p} \bar{\kappa}}$ are the *Schubert cells*.

Remarks 3.13. (1) The Schubert cells can be indexed by $W_K \backslash \widetilde{W} / W_K$ by Proposition 3.5 with the following in mind: Strictly speaking, using the Bruhat-Tits decomposition here, we arrive at something involving the Iwahori-Weyl group of $\underline{\mathcal{G}}_K \otimes \bar{\kappa}((t))$. However, by [PZ13, 9.2.2], this is isomorphic to the Iwahori-Weyl group of $G_{\breve{\mathbb{Q}}_p}$.

(2) $M_K^{\mathrm{loc}} \otimes \bar{\kappa}$ is a union of Schubert cells, namely of those indexed by $\mathrm{KR}(K, \{\mu\}) := W_K \backslash (W_K \mathrm{Adm}(\{\mu\}) W_K)/W_K$, cf. [PZ13, Theorem 9.3].

Remark 3.14. By construction, M_K^{loc} has an action b of $\underline{\mathcal{G}}_K \otimes_{\mathbb{Z}_p[t], t \mapsto p} \mathcal{O}_E \cong \mathcal{G}_K \otimes \mathcal{O}_E$.

For $w \in \widetilde{W}$ choose a representative $\dot{w} \in L\underline{\mathcal{G}}_K(\bar{\kappa})$ (with Remark 3.13 (1) in mind) and let $e_0 \in \mathrm{Gr}_{\underline{\mathcal{G}}_K \otimes_{\mathbb{Z}_p} \kappa}$ be the distinguished base point (associated with the identity). For $w \in W_K \mathrm{Adm}(\{\mu\}) W_K$, the orbit map of $\dot{w} \cdot e_0$ for the action a factors through the homomorphism $L^+(\underline{\mathcal{G}}_K \otimes \bar{\kappa}[[t]]) \to \mathcal{G}_K \otimes \kappa \cong \underline{\mathcal{G}}_K \otimes \bar{\kappa}$.

The orbits associated with the two $\mathcal{G}_K \otimes \kappa$-actions a and b on $M_K^{\mathrm{loc}} \otimes \kappa$ agree. The orbits of the $\mathcal{G}_K \otimes \kappa$-action on $M_K^{\mathrm{loc}} \otimes \kappa$ are indexed by $\mathrm{KR}(K, \{\mu\})$.

Definition 3.15. The stratifications thus obtained on $M_K^{\mathrm{loc}} \otimes \kappa$ and $\mathscr{S}_K \otimes \kappa$ are called *Kottwitz-Rapoport stratifications*. That is to say that Kottwitz-Rapoport strata on $\mathscr{S}_K \otimes \kappa$ are by definition pullbacks of Kottwitz-Rapoport strata on M_K^{loc}, which in turn are $\mathcal{G}_K \otimes \kappa$-orbits.

3.1.3 Ekedahl-Oort stratification

The Ekedahl-Oort stratification is only defined in the case of good reduction, i.e., if K_p is hyperspecial or, equivalently, if $\mathcal{G}_K$ is a *reductive* model of $G_{\mathbb{Q}_p}$. Then $G_{\mathbb{Q}_p}$ splits over $\breve{\mathbb{Q}}_p$ (by definition of “hyperspecial”, cf. [Tit79, 1.10.2]).

We therefore put ourselves in the situation of good reduction for this subsection.

Remark 3.16. Then W as defined in Definition 3.3 (1) agrees with the absolute Weyl group of $G_{\mathbb{Q}_p} = \mathcal{G}_K \otimes \mathbb{Q}_p$, which in turn agrees with the absolute Weyl group of $\overline{\mathcal{G}}_K := \mathcal{G}_K \otimes \kappa$, cf. [VW13, App. A.5].

Definition 3.17. Define I to be the type (interpreted as a subset of simple reflections) of the parabolic subgroup of $G_{\mathbb{Q}_p}$ defined by μ^{-1} (cf. Remark 1.73), and ${}^IW \subseteq W$ to be the system of representatives of the quotient group $W_I \backslash W$ containing the element of least length of every coset.

Theorem 3.18. [MW04; PWZ15; PWZ11; Zha18] *There is a smooth algebraic stack $\overline{\mathcal{G}}_K\text{-Zip}_\kappa := \overline{\mathcal{G}}_K\text{-Zip}_\kappa^\mu$ over κ with underlying topological space ${}^IW$ together with a smooth morphism*

$$\mathscr{S}_K \otimes \kappa \to \overline{\mathcal{G}}_K\text{-Zip}_\kappa^\mu.$$

The stratification of $\mathscr{S}_K \otimes \kappa$ thus obtained is the Ekedahl-Oort stratification.

3.1.4 EKOR stratification

Definition 3.19. Let L be a valued field extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, maximal ideal $\mathfrak{m}$ and residue field λ.

The *pro-unipotent radical* of $\mathcal{G}_K(\mathcal{O})$ is

$$\mathcal{G}_K(\mathcal{O})_1 := \{g \in \mathcal{G}_K(\mathcal{O}) \mid (g \mod \mathfrak{m}) \in \bar{R}_K(\lambda)\},$$

where $\bar{R}_K$ is the unipotent radical of $\mathcal{G}_K \otimes_{\mathbb{Z}_p} \lambda$.

In particular, if K is hyperspecial, then $\mathcal{G}_K(\mathcal{O})_1 = \ker(\mathcal{G}_K(\mathcal{O}) \to \mathcal{G}_K(\lambda))$.

Also, $\overline{\breve{K}} := \breve{K}/\breve{K}_1 \cong \overline{\mathcal{G}}_K^{\mathrm{rdt}}(\bar{\mathbb{F}}_p)$, where $\overline{\mathcal{G}}_K^{\mathrm{rdt}}$ is the maximal reductive quotient of $\overline{\mathcal{G}}_K := \mathcal{G}_K \otimes \kappa$.

Remark 3.20. [HR17, after Cor. 6.2] We have a commutative diagram

$$\begin{tikzcd} G(\breve{\mathbb{Q}}_p)/\breve{K}_\sigma(\breve{K}_1 \times \breve{K}_1) \arrow[r] \arrow[d] & {}^K\widetilde{W} \arrow[d] \\ \breve{K}\backslash G(\breve{\mathbb{Q}}_p)/\breve{K} \arrow[r] & W_K\backslash\widetilde{W}/W_K. \end{tikzcd}$$

Consider the map

$$\upsilon_K \colon \mathscr{S}_K \otimes \kappa \to G(\breve{\mathbb{Q}}_p)/\breve{K}_\sigma(\breve{K}_1 \times \breve{K}_1),$$

which is the composition of $\Upsilon_K \colon \mathscr{S}_K \otimes \kappa \to G(\breve{\mathbb{Q}}_p)/\breve{K}_\sigma$ (defined in Definition 2.4) with the projection $G(\breve{\mathbb{Q}}_p)/\breve{K}_\sigma \to G(\breve{\mathbb{Q}}_p)/\breve{K}_\sigma(\breve{K}_1 \times \breve{K}_1)$. The Kottwitz-Rapoport map $\lambda_K \colon \mathscr{S}_K \otimes \kappa \to \breve{K}\backslash G(\breve{\mathbb{Q}}_p)/\breve{K}$ factors through this map.

The fibers of υ_K are called *EKOR strata*. By [HR17, Thm. 6.15], they are locally closed subsets of $\mathscr{S}_K \otimes \kappa$.

Remarks 3.21. (1) One can explicitly express the image of a EKOR stratum under a change-of-parahoric map as a union of EKOR strata on the target [HR17, Prop. 6.11].

(2) The closure of an EKOR stratum is a union of EKOR strata and one can explicitly describe the associated order relation [HR17, Thm. 6.15].

Remark 3.22. In the hyperspecial case, the EKOR stratification agrees with the Ekedahl-Oort stratification. In the Iwahori case, it agrees with the Kottwitz-Rapoport stratification (${}^K\widetilde{W} = \widetilde{W} = W_K\backslash\widetilde{W}/W_K$ in that case).

By definition, the EKOR stratification always is a refinement of the Kottwitz-Rapoport stratification. So one way of approaching the EKOR stratification is by looking at a fixed Kottwitz-Rapoport stratum and trying to understand how it is subdivided into EKOR strata.

To get this started, let us recall some calculations from the proof of [HR17, Thm. 6.1].

Fixing a Kottwitz-Rapoport stratum means restricting our view to $\breve{K}w\breve{K}/\breve{K}_\sigma$ rather than the whole of $G(\breve{\mathbb{Q}}_p)/\breve{K}_\sigma$, for some fixed $w \in \mathrm{KR}(K, \{\mu\})$. The EKOR strata in the Kottwitz-Rapoport stratum associated with w are therefore indexed by $\breve{K}w\breve{K}/\breve{K}_\sigma(\breve{K}_1 \times \breve{K}_1)$.

Define $\sigma' := \sigma \circ \mathrm{Ad}(w)$ and consider the bijection

$$\begin{aligned} \breve{K}/(\breve{K} \cap w^{-1}\breve{K}w)_{\sigma'} &\xrightarrow{\sim} \breve{K}w\breve{K}/\breve{K}_\sigma, \\ k &\mapsto wk, \\ k_2\sigma(k_1) &\leftarrow\!\shortmid\, k_1wk_2. \end{aligned}$$

Let J be the set of simple affine reflections in W_K, let $\bar{B}$ be the image of $\breve{I}$ in $\overline{\breve{K}}$ and $\bar{T} \subseteq \bar{B}$ the maximal torus. Set $J_1 := J \cap w^{-1}Jw$.

Proposition 3.23. (See [Mor93, Lemma 3.19].) *The image of $\breve{K} \cap w^{-1}\breve{K}w$ in $\overline{\breve{K}}$ is $\bar{P}_{J_1}$, i.e., the standard parabolic subgroup of $\overline{\breve{K}}$ associated with J_1.*

Remark 3.24. He and Rapoport invoke Carter's book [Car93] at this point, which primarily pertains to the case of (usual) BN-pairs attached to reductive groups. Morris [Mor93] shows that the relevant results carry over likewise to the case of generalized (or affine) BN-pairs.

Then we get a map

$$\begin{aligned} \breve{K}w\breve{K}/\breve{K}_\sigma \to \breve{K}/(\breve{K} \cap w^{-1}\breve{K}w)_{\sigma'} &\to \overline{\breve{K}}/(\bar{P}_{J_1})_{\sigma'} \\ &\to \overline{\breve{K}}/(\bar{L}_{J_1})_{\sigma'}(\bar{U}_{J_1})_{\sigma'} \to \overline{\breve{K}}/(\bar{L}_{J_1})_{\sigma'}(\bar{U}_{J_1} \times \bar{U}_{\sigma'(J_1)}), \end{aligned}$$

which factors through a bijection

$$\breve{K}w\breve{K}/\breve{K}_\sigma(\breve{K}_1 \times \breve{K}_1) \xrightarrow{\sim} \overline{\breve{K}}/(\bar{L}_{J_1})_{\sigma'}(\bar{U}_{J_1} \times \bar{U}_{\sigma'(J_1)}) \cong \overline{\mathcal{G}}_K^{\mathrm{rdt}}\text{-Zip}^{\mathcal{Z}_w}(\bar{\mathbb{F}}_p)/\cong.$$

Here, $\mathcal{Z}_w$ is the (connected) algebraic zip datum $\mathcal{Z}_w = (\overline{\mathcal{G}}^{\mathrm{rdt}}, \bar{P}_{J_1}, \bar{P}_{\sigma'(J_1)}, \sigma')$, as described in [SYZ19]. In [SYZ19], Shen, Yu and Zhang show that this observation "globalizes" (with the drawback that "global" here still just refers to the Kottwitz-Rapoport stratum[3]) in a pleasant way. To wit, one gets a smooth morphism [SYZ19, Theorem A]

$$\zeta_w \colon \overline{\mathscr{S}}_K^w \to \overline{\mathcal{G}}_K^{\mathrm{rdt}}\text{-Zip}_\kappa^{\mathcal{Z}_w}$$

(the source being a Kottwitz-Rapoport stratum).

[3]They also give another "globalization"; the drawback there being that it only works after perfection.

3.2 $\overline{\mathcal{G}}_K$-zips in the Siegel case

Here we work with the Siegel Shimura datum, cf. Example 1.2.

3.2.1 Preliminaries

Notation 3.25. Fix $p \neq 2$,[4] $g \in \mathbb{Z}_{\geq 1}$ and a subset $J \subseteq \mathbb{Z}$ with $J = -J$ and $J + 2g\mathbb{Z} = J$. Associated with J is the partial lattice chain $\{\Lambda^j \mid j \in J\}$, where Λ^j are defined as in equation (1.69). Let K_p be the corresponding parahoric subgroup of $\mathrm{GSp}_{2g}(\mathbb{Q}_p)$, i.e., the stabilizer of said lattice chain. It contains the Iwahori subgroup I_p associated with the full lattice chain (1.69). For the maximal torus T we take the usual diagonal (split) torus.

Remark 3.26. The Weyl group is

$$\begin{aligned} W &= \{\pi \in S_{2g} = \mathrm{Aut}(\{\pm 1, \pm 2, \dots, \pm g\}) \mid \pi(-n) = -\pi(n) \text{ for } n = 1, 2, \dots, g\} \\ &\cong S_g \ltimes \{\pm 1\}^g. \end{aligned}$$

Here the transposition $(n \quad m)$ of $S_g = \mathrm{Aut}(\{1, 2, \dots, g\})$ corresponds to the element $(n \quad m)(-n \quad -m)$ of $\mathrm{Aut}(\{\pm 1, \pm 2, \dots, \pm g\})$ and the element of $\{\pm 1\}^g$ which has a -1 in position i and 1 everywhere else corresponds to $(i \quad -i)$.

The affine Weyl group is $W_a = W \ltimes Y_0$ and the Iwahori-Weyl group $\widetilde{W} = W \ltimes Y$ with

$$\begin{aligned} \mathbb{Z}^{g+1} \cong Y &= \{(\nu_1, \dots, \nu_g, \nu_{-g}, \dots, \nu_{-1}) \in \mathbb{Z}^{2g} : \nu_1 + \nu_{-1} = \dots = \nu_g + \nu_{-g}\} \\ \supseteq Y_0 &= \{(\nu_1, \dots, \nu_g, \nu_{-g}, \dots, \nu_{-1}) \in \mathbb{Z}^{2g} : 0 = \nu_1 + \nu_{-1} = \dots = \nu_g + \nu_{-g}\} \cong \mathbb{Z}^g. \end{aligned}$$

The simple affine roots (whose walls bound the base alcove $\mathfrak{a}$) are

$$\begin{aligned} &1 - 2e_{-1} + e_0 = 1 + 2e_1 - e_0, \\ &e_{-1} - e_{-2} = e_2 - e_1, e_{-2} - e_{-3}, \dots, e_{-(g-1)} - e_{-g}, \\ &2e_{-g} - e_0 = e_0 - 2e_g, \end{aligned}$$

where $e_1, \dots, e_g, e_{-g}, \dots, e_{-1} \colon T \to \mathbb{G}_m$ are the obvious cocharacters and $e_0 = e_1 + e_{-1} = \dots = e_g + e_{-g}$.

[4] As in [RZ96], the principal reason for this restriction is our use of the equivalence between alternating and skew-symmetric. See Definition 3.30 (e).

The reflections corresponding to the simple affine roots are

$$((1 \quad -1), \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}), (-1 \quad -2)(1 \quad 2), \dots, (-g \quad -(g-1))(g \quad g-1), (g \quad -g).$$

The length zero subgroup $\Omega \subseteq \widetilde{W}$ is generated by $((w_0, \epsilon), y) \in (S_g \ltimes \{\pm 1\}^g) \ltimes Y$, where $w_0 \in S_g$ is the longest element, $\epsilon = (-1, -1, \dots, -1)$ and $y = (0^g, 1^g)$.

Remark 3.27. One also can choose $\cdots \subseteq p\mathbb{Z}_p \oplus \mathbb{Z}_p^{2g-1} \subseteq \mathbb{Z}_p^{2g} \subseteq \cdots$ instead of $\cdots \subseteq \mathbb{Z}_p^{2g-1} \oplus p\mathbb{Z}_p \subseteq \mathbb{Z}_p^{2g} \subseteq \cdots$ as the standard lattice chain. Then the simple affine roots would be

$$1 - 2e_1 + e_0, e_1 - e_2 = e_2 - e_1, e_2 - e_3, \dots, e_{g-1} - e_g, 2e_g - e_0.$$

Remark 3.28. $\widetilde{W} = W \ltimes Y = N(\mathbb{Q}_p)/T(\mathbb{Z}_p)$ and $N(\mathbb{Q}_p) \to W \ltimes Y$ has a section $W \ltimes Y \to N(\mathbb{Q}_p)$, which sends $(\pi, \underline{\nu}) \in W \ltimes Y$ to $T_{\underline{\nu}} P_w$, where $T_{\underline{\nu}} = \begin{pmatrix} p^{\nu_1} & & & & \\ & p^{\nu_2} & & & \\ & & \ddots & & \\ & & & p^{\nu_{-2}} & \\ & & & & p^{\nu_{-1}} \end{pmatrix}$ and P_w is the permutation matrix with $P_w(e_i) = e_{w(i)}$.

Remark 3.29. Using the results of [KR00] we also easily can compute $\mathrm{Adm}(\{\mu\})$. One potential source of confusion at this point is that, due to our choice of the base alcove (cf. Remark 3.27), in our setup we need to use $\omega_i := (0^{2g-i}, 1^i)$ instead of $\omega_i := (1^i, 0^{2g-i})$ (notation of [KR00]), cf. [Yu08, p. 1268]. With that convention in place, we have that $x \in \widetilde{W}$ is $\{\mu\}$-admissible ($\mu = (1^g, 0^g)$) if and only if

$$(0, \dots, 0) \leq x(\omega_i) - \omega_i \leq (1, \dots, 1) \quad \text{for all } 0 \leq i < 2g$$

(component-wise comparison).

3.2.2 Lattice chains, zips, admissibility

Definition 3.30. Let S be a $\mathbb{Z}_p$-scheme.

A *Siegel lattice chain in the weak sense on S of type J* is a tuple $(\mathcal{V}^\bullet, \mathcal{L}, \alpha_{\bullet\bullet}, \theta_\bullet, \psi_\bullet)$, where

(a) for all $j \in J$, $\mathcal{V}^j$ is a vector bundle on S of rank $2g$,

(b) $\mathcal{L}$ is a line bundle on S,

(c) for all $i, j \in J$ with $j > i$, $\alpha_{j,i}\colon \mathcal{V}^j \to \mathcal{V}^i$ is a vector bundle homomorphism, such that the $(\alpha_{j,i})$ satisfy the obvious cocycle condition (and we also define $\alpha_{i,i} := \mathrm{id}$),

(d) for all $j \in J$, $\theta_j\colon \mathcal{V}^j \xrightarrow{\sim} \mathcal{V}^{j-2g}$ is a vector bundle isomorphism such that the (θ_j) are compatible with the $(\alpha_{j,i})$ in that $\theta_i \circ \alpha_{j,i} = \alpha_{j-2g,i-2g} \circ \theta_j$ and $\alpha_{j,j-2g} = p \cdot \theta_j$,

(e) for all $j \in J$ a vector bundle isomorphism $\psi_j\colon \mathcal{V}^j \xrightarrow{\sim} (\mathcal{V}^{-j})^* \otimes \mathcal{L}$ compatible with (θ_j) and $(\alpha_{j,i})$, such that $-\psi_j(x, y) = \psi_{-j}(y, x)$ for all $(x, y) \in \mathcal{V}^j \times \mathcal{V}^{-j}$.[5]

We also have a *standard* Siegel lattice chain in the weak sense on $\operatorname{Spec} \mathbb{Z}_p$ (and hence by base change on every $\mathbb{Z}_p$-scheme S) of type J, namely the one given by the lattice chain $\{\Lambda^j \mid j \in J\}$. We can think of the standard Siegel lattice chain as either having varying $\mathcal{V}^j$ with the $\alpha_{j,i}$ being the obvious inclusion maps (e.g. (if $\{0, 1\} \subseteq J$), $\mathcal{V}^1 = \mathbb{Z}_p^{2g-1} \oplus p\mathbb{Z}_p \xrightarrow{\alpha_{1,0}=\text{inclusion}} \mathbb{Z}_p^{2g} = \mathcal{V}^0$) or as having constant $\mathcal{V}^j = \mathbb{Z}_p^{2g}$ with the $\alpha_{j,i}$ being diagonal matrices with all entries either p or 1 (e.g., $\mathcal{V}^1 = \mathbb{Z}_p^{2g} \xrightarrow{\alpha_{1,0}=\mathrm{diag}(1,1,\dots,1,p)} \mathbb{Z}_p^{2g} = \mathcal{V}^0$). Usually the latter point of view is more convenient.

A *Siegel lattice chain on S of type J* (or *Siegel lattice chain in the strong sense on S of type J*) then is a Siegel lattice chain in the weak sense on S of type J that Zariski-locally on S is isomorphic to the standard chain.

Remarks 3.31. (1) Let $(\mathcal{V}^\bullet, \mathcal{L}, \alpha_{\bullet\bullet}, \theta_\bullet, \psi_\bullet)$ be a Siegel lattice chain in the weak sense on S of type J. Then $\tilde{\psi}_j := (\tilde{\alpha}_{j,-j}^* \otimes \mathrm{id}_{\mathcal{L}}) \circ \psi_j\colon \mathcal{V}^j \otimes \mathcal{V}^j \to \mathcal{L}$ is alternating.

Here $\tilde{\alpha}_{j,-j}$ is defined as follows: Let $n \in \mathbb{Z}$ be maximal with $j - 2gn \geq -j$. Then $\tilde{\alpha}_{j,-j} := \alpha_{j-2gn,-j} \circ \theta_{j-2g(n-1)} \circ \dots \circ \theta_j$.

(2) Note that this means that $\tilde{\psi}_j$ is (twisted) symplectic if $-j \in j + 2g\mathbb{Z}$, i.e., if $j \in g\mathbb{Z}$.

[5]By "$(x, y) \in \mathcal{V}^j \times \mathcal{V}^{-j}$" we of course mean that there is an open subset $U \subseteq S$ such that $(x, y) \in (\mathcal{V}^j \times \mathcal{V}^{-j})(U)$.

PROOF: (of (1)) Let $x, y \in \mathcal{V}^j$. Then

$$\begin{aligned}
\tilde{\psi}_j(x,y) &= \psi_j(x, \tilde{\alpha}_{j,-j}(y)) \\
&= \psi_j(x, (\alpha_{j-2gn,-j} \circ \theta_{j-2g(n-1)} \circ \cdots \circ \theta_j)(y)) \\
&= \psi_{2gn-j}(\alpha_{j,2gn-j}(x), (\theta_{j-2g(n-1)} \circ \cdots \circ \theta_j)(y)) \\
&= \psi_{2g(n-1)-j}((\theta_{2gn-j} \circ \alpha_{j,2gn-j})(x), (\theta_{j-2g(n-2)} \circ \cdots \circ \theta_j)(y)) \\
&= \cdots \\
&= \psi_{-j}((\theta_{-j+2g} \circ \cdots \circ \theta_{2gn-j} \circ \alpha_{j,2gn-j})(x), y) \\
&= -\psi_j(y, (\theta_{-j+2g} \circ \cdots \circ \theta_{2gn-j} \circ \alpha_{j,2gn-j})(x)) \\
&= -\tilde{\psi}_j(y,x).
\end{aligned}$$

□

Reminder 3.32. $\mathcal{G}_K$ is the automorphism group of the standard Siegel lattice chain.

The following definition is a generalization of [VW13, Definition 3.1] in the Siegel case.

Definition 3.33. Let S be an $\mathbb{F}_p$-scheme.
A $\overline{\mathcal{G}}_K$-zip over S is a tuple $(\mathcal{V}^\bullet, \mathcal{L}, \alpha_{\bullet\bullet}, \theta_\bullet, \psi_\bullet, \mathcal{C}^\bullet, \mathcal{D}^\bullet, \varphi_0^\bullet, \varphi_1^\bullet, \varphi_{\mathcal{L}})$, where

(a) $(\mathcal{V}^\bullet, \mathcal{L}, \alpha_{\bullet\bullet}, \theta_\bullet, \psi_\bullet)$ is a Siegel lattice chain on S of type J,

(b) for all $j \in J$, $\mathcal{C}^j \subseteq \mathcal{V}^j$ are locally direct summands of rank g compatible with $\alpha_{\bullet\bullet}, \theta_\bullet$, such that

$$\mathcal{C}^j \hookrightarrow \mathcal{V}^j \overset{\psi_j}{\cong} (\mathcal{V}^{-j})^* \otimes \mathcal{L} \to (\mathcal{C}^{-j})^* \otimes \mathcal{L}$$

vanishes. (cf. Remark 1.70 for the origins of this condition.)

(c) $\mathcal{D}^\bullet \subseteq \mathcal{V}^\bullet$ satisfies the same conditions as $\mathcal{C}^\bullet \subseteq \mathcal{V}^\bullet$,

(d) $\varphi_0^j \colon (\mathcal{C}^j)^{(p)} \xrightarrow{\sim} \mathcal{V}^j/\mathcal{D}^j$ and $\varphi_1^j \colon (\mathcal{V}^j/\mathcal{C}^j)^{(p)} \xrightarrow{\sim} \mathcal{D}^j$ are isomorphisms of vector bundles compatible with $\alpha_{\bullet\bullet}$ and $\theta_\bullet$ and $\varphi_{\mathcal{L}} \colon \mathcal{L}^{(p)} \xrightarrow{\sim} \mathcal{L}$ is an isomorphism of line bundles, such that

$$\begin{tikzcd}
(\mathcal{C}^j)^{(p)} \arrow[r, "\psi_j^{(p)}"] \arrow[d, "\varphi_0^j"] & (\mathcal{V}^{-j}/\mathcal{C}^{-j})^{*,(p)} \otimes \mathcal{L}^{(p)} \\
\mathcal{V}^j/\mathcal{D}^j \arrow[r, "\psi_j"] & (\mathcal{D}^{-j})^* \otimes \mathcal{L} \arrow[u, "(\varphi_1^{-j})^* \otimes \varphi_{\mathcal{L}}^{-1}"']
\end{tikzcd}$$

commutes, i.e.,

$$\psi_j(\varphi_0^j(\_), \varphi_1^{-j}(\_)) = \varphi_{\mathcal{L}} \circ \psi_j^{(p)}(\_,\_)\colon (\mathcal{C}^j)^{(p)} \times (\mathcal{V}^{-j}/\mathcal{C}^{-j})^{(p)} \to \mathcal{L}^{(p)} \to \mathcal{L}.$$

Since $\varphi_{\mathcal{L}}$ evidently is uniquely determined by the other data, we sometimes leave it out. We obtain a fibered category $\overline{\mathcal{G}}_K\text{-Zip} \to \mathrm{Sch}_{\mathbb{F}_p}$.

Remark 3.34. ψ_j gives rise to isomorphisms

$$\begin{aligned} \mathcal{C}^j &\xrightarrow{\sim} (\mathcal{V}^{-j}/\mathcal{C}^{-j})^* \otimes \mathcal{L}, \\ \mathcal{V}^j/\mathcal{C}^j &\xrightarrow{\sim} (\mathcal{C}^{-j})^* \otimes \mathcal{L}, \\ \mathcal{D}^j &\xrightarrow{\sim} (\mathcal{V}^{-j}/\mathcal{D}^{-j})^* \otimes \mathcal{L}, \\ \mathcal{V}^j/\mathcal{D}^j &\xrightarrow{\sim} (\mathcal{D}^{-j})^* \otimes \mathcal{L}. \end{aligned}$$

This way $\mathcal{V}^\bullet/\mathcal{C}^\bullet \oplus \mathcal{C}^\bullet$ and $\mathcal{D}^\bullet \oplus \mathcal{V}^\bullet/\mathcal{D}^\bullet$ become Siegel lattice chains in the weak(!) sense of type J. The Cartier isomorphism then is an isomorphism in the category of Siegel lattice chains in the weak sense of type J. Over an algebraically closed field, we call the isomorphism type of the Siegel lattice chain in the weak sense $\mathcal{V}^\bullet/\mathcal{C}^\bullet \oplus \mathcal{C}^\bullet$ the *Kottwitz-Rapoport type of the $\overline{\mathcal{G}}_K$-zip*.

We also define a linearly rigidified version of $\overline{\mathcal{G}}_K\text{-Zip}$ as follows.

Definition 3.35. We define the fibered category $\overline{\mathcal{G}}_K\text{-Zip}^\sim \to \mathrm{Sch}_{\mathbb{F}_p}$ just like $\overline{\mathcal{G}}_K\text{-Zip}$ but with the extra condition that $(\mathcal{V}^\bullet, \mathcal{L}, \alpha_{\bullet\bullet}, \theta_\bullet, \psi_\bullet)$ be the standard Siegel lattice chain (rather than just locally isomorphic to it).

Lemma 3.36. *We always have a closed embedding of $\overline{\mathcal{G}}_K\text{-Zip}^\sim$ into a product of (classical) $\mathrm{GL}_{2g}\text{-Zip}^\sim$'s, and therefore $\overline{\mathcal{G}}_K\text{-Zip}^\sim$ is a scheme.*

Proof: Set $J' := J \cap \{0, \dots, 2g-1\}$. Let $\mathrm{GL}_{2g}\text{-Zip}^\sim = E_{(1^g,0^g)}\backslash(\mathrm{GL}_{2g} \times \mathrm{GL}_{2g})$ be the $\mathbb{F}_p$-scheme of trivialized GL_{2g}-zips (so that $[\mathrm{GL}_{2g} \backslash \mathrm{GL}_{2g}\text{-Zip}^\sim] = \mathrm{GL}_{2g}\text{-Zip}$) with respect to the the cocharacter $(1^g, 0^g)$, and $\prod_{j\in J'} \mathrm{GL}_{2g}\text{-Zip}^\sim$ the product of $\#J'$ copies of this scheme. On J' we define $-j := 2g - j$ for $1 \le j \le 2g-1$ and $-0 := 0$.

Then we get a monomorphism

$$\overline{\mathcal{G}}_K\text{-Zip}^\sim \hookrightarrow \prod_{j\in J'} \mathrm{GL}_{2g}\text{-Zip}^\sim \tag{3.37}$$

by sending $(\mathcal{C}^\bullet, \mathcal{D}^\bullet, \varphi_0^\bullet, \varphi_1^\bullet)$ to $\left(\mathcal{C}^j, \mathcal{D}^j, \varphi_0^j, \varphi_1^j\right)_{j \in J'}$.

The extra conditions for an element of $\prod_{j \in J'} \mathrm{GL}_{2g}\text{-Zip}^{\sim}$ to be in $\overline{\mathcal{G}}_K\text{-Zip}^{\sim}$ are as in Definition 3.33:

(1) $\mathcal{C}^\bullet, \mathcal{D}^\bullet, \varphi_0^\bullet, \varphi_1^\bullet$ are compatible with the transition maps (or, to put it differently, $(\mathcal{C}^j \oplus \mathcal{V}^j/\mathcal{C}^j)^{(p)} \xrightarrow[\cong]{\varphi_0^j \oplus \varphi_1^j} \mathcal{V}^j/\mathcal{D}^j \oplus \mathcal{D}^j$ is compatible with the transition maps),

(2) $\mathcal{C}^j \hookrightarrow \mathcal{V}^j \overset{\psi_j}{\cong} (\mathcal{V}^{-j})^* \to (\mathcal{C}^{-j})^*$ vanishes.

(3) $\mathcal{D}^j \hookrightarrow \mathcal{V}^j \overset{\psi_j}{\cong} (\mathcal{D}^{-j})^* \to (\mathcal{D}^{-j})^*$ vanishes.

(4) There is a (necessarily unique) isomorphism $\varphi_{\mathcal{L}} \colon \mathcal{L}^{(p)} \xrightarrow{\sim} \mathcal{L} = \mathcal{O}_S$ of line bundles, such that

$$\begin{tikzcd}
(\mathcal{C}^j)^{(p)} \arrow[r, "\psi_j^{(p)}"] \arrow[d, "\varphi_0^j"] & (\mathcal{V}^{-j}/\mathcal{C}^{-j})^{*,(p)} \\
\mathcal{V}^j/\mathcal{D}^j \arrow[r, "\psi_j"] & (\mathcal{D}^{-j})^* \arrow[u, "(\varphi_1^{-j})^* \otimes \varphi_{\mathcal{L}}^{-1}"']
\end{tikzcd}$$

commutes.

We claim that the conditions are closed on $\prod_{j \in J'} \mathrm{GL}_{2g}\text{-Zip}^{\sim}$ (hence the monomorphism is a closed immersion).

To see this, we recall the construction of the scheme $\mathrm{GL}_{2g}\text{-Zip}^{\sim}$ as executed in [MW04, (3.10), (3.11), (4.3)].

Recall our notational convention regarding the parabolic subgroup associated with a cocharacter χ from Definition 1.73. As in [MW04], we denote by Par_χ the scheme of parabolic subgroups of type χ.

There is a group scheme H defined by the cartesian diagram

$$\begin{tikzcd} H \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{P}_{((-1)^g,0^g)}/\mathcal{U}_{((-1)^g,0^g)} \arrow[d] \\ \mathrm{Par}_{((-1)^g,0^g)} \times \mathrm{Par}_{(1^g,0^g)} \arrow[r, "(\ )^{(p)} \circ \mathrm{pr}_1"] & \mathrm{Par}_{((-1)^g,0^g)} \end{tikzcd}$$

where $\mathcal{P}_{((-1)^g,0^g)} \to \mathrm{Par}_{((-1)^g,0^g)}$ is the universal parabolic group scheme and $\mathcal{U}_{((-1)^g,0^g)}$ its unipotent radical, such that $\mathrm{GL}_{2g}\text{-Zip}^{\sim}$ is an H-Zariski torsor over $\mathrm{Par}_{((-1)^g,0^g)} \times \mathrm{Par}_{(1^g,0^g)}$, where $\mathrm{GL}_{2g}\text{-Zip}^{\sim} \to \mathrm{Par}_{((-1)^g,0^g)} \times \mathrm{Par}_{(1^g,0^g)}$ is given by $(C, D, \varphi_0, \varphi_1) \mapsto (C, D)$.

Clearly, compatibility of $\mathcal{C}^\bullet, \mathcal{D}^\bullet$ with the transition maps is a closed condition on $\prod_{j\in J'} \mathrm{Par}_{((-1)^g,0^g)} \times \mathrm{Par}_{(1^g,0^g)}$ and then also on $\prod_{j\in J'} \mathrm{GL}_{2g}\text{-Zip}^{\sim}$. Similar for the conditions (2) and (3).

Locally, we can choose complements (not necessarily compatible with the transition maps) and then $\varphi_\bullet^j$ yield sections g^j of GL_{2g} as in [MW04, definition of $g \in G(S)$ in the proof of (4.3)]. The g^j are well-defined up to $\mathcal{U}^{(p)}_{((-1)^g,0^g)} \times \mathcal{U}_{(1^g,0^g)}$, and we want them to be compatible with the transition maps coming from the Siegel lattice chains in the weak sense $\mathcal{C}^j \oplus \mathcal{V}^j/\mathcal{C}^j$ and $\mathcal{V}^j/\mathcal{D}^j \oplus \mathcal{D}^j$, respectively. With our complements in place, these transition maps correspond to maps $\mathcal{V}^j \to \mathcal{V}^{j-n}$. The question of whether g^j is compatible with these maps is independent of the choice of complements (basically because the transition maps $\mathcal{V}^j \to \mathcal{V}^{j-n}$ depend on the choice of complements similar to how g^j depends on that choice).

So in effect we can view the conditions on $\varphi_0^\bullet, \varphi_1^\bullet$ of (1) as closed conditions on $\prod_{j\in J'} \mathrm{GL}_{2g}\text{-Zip}^{\sim\sim}$, where $\mathrm{GL}_{2g}\text{-Zip}^{\sim\sim} \to \mathrm{GL}_{2g}\text{-Zip}^{\sim}$ (an fpqc quotient map) additionally comes with complementary spaces of C, D ($\mathrm{GL}_{2g}\text{-Zip}^{\sim\sim} = \tilde{X}_\tau$ in the notation of [MW04, proof of (4.3)]).

We also can reformulate condition (4) in those terms. $\square$

Corollary 3.38. *$\overline{\mathcal{G}}_K$-Zip is the algebraic quotient stack $[\overline{\mathcal{G}}_K \backslash \overline{\mathcal{G}}_K\text{-Zip}^{\sim}]$.*

Here by definition an element $\phi \in \overline{\mathcal{G}}_K$ acts on $\overline{\mathcal{G}}_K$-Zip$^{\sim}$ by replacing $(\mathcal{C}^\bullet, \mathcal{D}^\bullet, \varphi_0^\bullet, \varphi_1^\bullet, \varphi_{\mathcal{L}})$ by $(\phi\mathcal{C}^\bullet, \phi\mathcal{D}^\bullet, \phi\varphi_0^\bullet\phi^{-(p)}, \phi\varphi_1^\bullet\phi^{-(p)}, \varphi_{\mathcal{L}})$.

Definition 3.39. We let an element $(X,Y) \in \overline{\mathcal{G}}_K \times \overline{\mathcal{G}}_K$ act on $\overline{\mathcal{G}}_K$-Zip by replacing $(\mathcal{V}^\bullet, \mathcal{L}, \alpha_{\bullet\bullet}, \theta_\bullet, \psi_\bullet, \mathcal{C}^\bullet, \mathcal{D}^\bullet, \varphi_0^\bullet, \varphi_1^\bullet)$ by

$$(\mathcal{V}^\bullet, \mathcal{L}, \alpha_{\bullet\bullet}, \theta_\bullet, \psi_\bullet, X\mathcal{C}^\bullet, Y\mathcal{D}^\bullet, Y\varphi_0^\bullet X^{-(p)}, Y\varphi_1^\bullet X^{-(p)}).$$

Notation 3.40. Let $\mathscr{S}_K \to \operatorname{Spec} \mathbb{Z}_p$ be the integral model of the Siegel Shimura variety of level K (where $K = K_pK^p$ with K^p sufficiently small), and recall $\tilde{\mathscr{S}}_K$ from Section 1.9.1. Moreover, define $\overline{\mathscr{S}}_K := \mathscr{S}_K \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ and $\overline{\tilde{\mathscr{S}}}_K := \tilde{\mathscr{S}}_K \otimes_{\mathbb{Z}_p} \mathbb{F}_p$. So $\overline{\tilde{\mathscr{S}}}_K \to \overline{\mathscr{S}}_K$ is a $\overline{\mathcal{G}}_K$-torsor.

Remark 3.41. We have morphisms $\overline{\tilde{\mathscr{S}}}_K \to \overline{\mathcal{G}}_K\text{-Zip}^\sim$ (take first de Rham cohomology with Frobenius and Verschiebung) and $\overline{\mathcal{G}}_K\text{-Zip}^\sim \to \overline{M}_K^{\mathrm{loc}}$ (take the $\mathcal{C}^\bullet$-filtration) and therefore

$$\overline{\mathscr{S}}_K \to \overline{\mathcal{G}}_K\text{-Zip} \to [\overline{\mathcal{G}}_K \backslash \overline{M}_K^{\mathrm{loc}}].$$

Remark 3.42. In particular, $\overline{\mathcal{G}}_K$-Zip has a Kottwitz-Rapoport stratification, which agrees with the notion of Kottwitz-Rapoport type as defined in Remark 3.34.

For $w \in \mathrm{KR}(K, \{\mu\})$ denote the associated Kottwitz-Rapoport stratum by $\overline{\mathcal{G}}_K\text{-Zip}_w$, i.e., we interpret w as a $\bar{\mathbb{F}}_p$-valued point of $[\overline{\mathcal{G}}_K \backslash \overline{M}_K^{\mathrm{loc}}]$ and form $\overline{\mathcal{G}}_K\text{-Zip}_w$ as a fiber product.

Construction 3.43. Fix $w \in \mathrm{Adm}(\{\mu\})^K \subseteq \widetilde{W}$ (so that $W_KwW_K \in \mathrm{KR}(K, \{\mu\})$). We define a standard $\overline{\mathcal{G}}_K$-zip of KR type W_KwW_K.

Using Remark 3.28, we interpret w as an element of $N(\mathbb{Q}_p) \subseteq G(\mathbb{Q}_p)$. The admissibility condition implies that we can interpret it as an endomorphism $w^\bullet$ of the standard lattice chain $\mathcal{V}^\bullet$ over $\mathbb{Z}_p$.[6]

We denote the standard Siegel lattice chain over $\mathbb{Z}_p$ by $\mathscr{V}^\bullet$ and its base change to $\mathbb{F}_p$ by $\mathcal{V}^\bullet$. Define $\mathscr{C}_w^\bullet := pw^{\bullet,-1}\mathscr{V}^\bullet$ and $\mathscr{D}_w^\bullet := \sigma(w^\bullet)\mathscr{V}^\bullet$. Then $\mathcal{C}_w^\bullet := \mathscr{C}_w^\bullet \otimes \mathbb{F}_p = \ker(w^\bullet \colon \mathcal{V}^\bullet \to \mathcal{V}^\bullet)$, so $(\mathcal{V}^\bullet/\mathcal{C}_w^\bullet)^{(p)} \xrightarrow{\sim} \mathcal{D}_w^\bullet := \mathscr{D}_w^\bullet \otimes \mathbb{F}_p$ via $\sigma(w^\bullet)$ and $(\mathcal{C}_w^\bullet)^{(p)} \xrightarrow{\sim} \mathcal{V}^\bullet/\mathcal{D}_w^\bullet$ via $p^{-1}\sigma(w^\bullet)$.

This defines a standard element $\widetilde{\mathrm{Std}}(w)$ of $\overline{\mathcal{G}}_K\text{-Zip}^\sim(\mathbb{F}_p)$ and a standard element $\mathrm{Std}(w)$ of $\overline{\mathcal{G}}_K\text{-Zip}_w(\mathbb{F}_p)$.

[6] Take up the second point of view described in Definition 3.30 regarding $\mathcal{V}^\bullet$. Define $\underline{\nu}^{(0)} := \underline{\nu}$, $\underline{\nu}^{(1)} := \underline{\nu} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ -1 \end{pmatrix} + w \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{pmatrix}$, $\underline{\nu}^{(2)} := \underline{\nu} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ -1 \end{pmatrix} + w \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 1 \end{pmatrix}$, and so on. Then $w^j = T_{\underline{\nu}^{(j)}} P_w$ for $0 \leq j < 2g$.

From the formulation of the admissibility condition as in Remark 3.29, we see that $w \in \mathrm{Adm}(\{\mu\})^K$ is equivalent to the condition that $\underline{\nu}^{(j)}$ be a permutation of $(1^g, 0^g)$ for all relevant j.

Definition and Remark 3.44. (See also [SYZ19, Lemma 3.3.2].) $\mathcal{G}_w := \mathrm{Aut}(\mathscr{C}_w^\bullet \subseteq \mathscr{V}^\bullet)$ is a Bruhat-Tits group scheme with generic fiber $G_{\mathbb{Q}_p}$ and $\breve{\mathbb{Z}}_p$-points $\breve{K} \cap w^{-1}\breve{K}w$; and similarly for $\mathcal{G}_{\sigma(w)^{-1}} := \mathrm{Aut}(\mathscr{D}_w^\bullet \subseteq \mathscr{V}^\bullet)$ with $\breve{K} \cap \sigma(w)\breve{K}\sigma(w)^{-1}$.

Definition 3.45. We keep $w \in \mathrm{Adm}(\{\mu\})^K \subseteq \widetilde{W}$ fixed and define $\widetilde{E}_w \subseteq \overline{\mathcal{G}}_K \times \overline{\mathcal{G}}_K$ to be the stabilizer of $\widetilde{\mathrm{Std}}(w)$.

So $\widetilde{E}_w$ consists of those $(X^\bullet, Y^\bullet) \in \overline{\mathcal{G}}_K \times \overline{\mathcal{G}}_K$ such that $X^\bullet \mathcal{C}_w^\bullet = \mathcal{C}_w^\bullet$, $Y^\bullet \mathcal{D}_w^\bullet = \mathcal{D}_w^\bullet$, and $Y^\bullet \circ \varphi_j^\bullet \circ X^{\bullet,-(p)} = \varphi_j^\bullet$ for $j = 0, 1$.

In the notation of [SYZ19, Lemma 3.3.2] we have

$$\widetilde{E}_w = \overline{\mathcal{G}}_w \times_{\overline{\mathcal{G}}_w^{L,(p)}} \overline{\mathcal{G}}_{\sigma(w)^{-1}}. \tag{3.46}$$

Here $\overline{\mathcal{G}}_w^L$ is the image of $\overline{\mathcal{G}}_w$ in $\mathrm{DiagAut}(\mathcal{C}_w^\bullet \oplus \mathcal{V}^\bullet/\mathcal{C}_w^\bullet)$ (the automorphisms of $\mathcal{C}_w^\bullet \oplus \mathcal{V}^\bullet/\mathcal{C}_w^\bullet$ respecting both $\mathcal{C}_w^\bullet$ and $\mathcal{V}^\bullet/\mathcal{C}_w^\bullet$).

The orbit of $\widetilde{\mathrm{Std}}(w)$ in $\overline{\mathcal{G}}_K\text{-Zip}^\sim$ is the fppf quotient $(\overline{\mathcal{G}}_K \times \overline{\mathcal{G}}_K)/\widetilde{E}_w$, cf. [DG80, II, § 5, no. 3].

Lemma 3.47. *We have commutative diagrams*

$$\begin{tikzcd}
\overline{\mathcal{G}}_w \arrow[r, hook] \arrow[d, two heads] & \overline{\mathcal{G}} \arrow[d, two heads] \\
\bar{P}_{J_1} \arrow[r, hook] & \overline{\mathcal{G}}^{\mathrm{rdt}}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\overline{\mathcal{G}}_{\sigma(w)^{-1}} \arrow[r, hook] \arrow[d, two heads] & \overline{\mathcal{G}} \arrow[d, two heads] \\
\bar{P}_{\sigma'(J_1)} \arrow[r, hook] & \overline{\mathcal{G}}^{\mathrm{rdt}}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\overline{\mathcal{G}}_w^L \arrow[r, hook] \arrow[d, two heads] & \overline{\mathcal{G}} \arrow[d, two heads] \\
\bar{L}_{J_1}. \arrow[r, hook] & \overline{\mathcal{G}}^{\mathrm{rdt}}
\end{tikzcd}$$

Proof: This follows from Proposition 3.23. $\square$

Lemma 3.48. *The image of $\widetilde{E}_w$ under $\overline{\mathcal{G}} \times \overline{\mathcal{G}} \to \overline{\mathcal{G}}^{\mathrm{rdt}} \times \overline{\mathcal{G}}^{\mathrm{rdt}}$ is $E_{\mathcal{Z}_w}$.*

Proof: This follows from Lemma 3.47. $\square$

Lemma 3.49. *Assume $0 \in J$. The $\overline{\mathcal{G}}_K \times \overline{\mathcal{G}}_K$-orbit of $\widetilde{\mathrm{Std}}(w)$ for $w \in \mathrm{Adm}(\{\mu\})^K$ depends only on $W_K w W_K$.*

Proof: Let $x, y \in W_K \subseteq W$. As above we get endomorphisms $x^\bullet, y^\bullet$ of $\mathcal{V}^\bullet$, which in this case are in fact automorphisms. Now $\widetilde{\mathrm{Std}}(w) = ((y^\bullet)^{-1}, \sigma(x^\bullet)) \cdot \widetilde{\mathrm{Std}}(w)$. $\square$

Definition 3.50. Define $\overline{\mathcal{G}}_K\text{-AdmZip}^\sim$ to be the union of the $\overline{\mathcal{G}}_K \times \overline{\mathcal{G}}_K$-orbits of the standard zips $\widetilde{\mathrm{Std}}(w)$ for $w \in \mathrm{Adm}(\{\mu\})^K$. Here an orbit by definition is the image of the orbit map endowed with the reduced subscheme structure, and—as we prove just below—the union of orbits just referred to is a closed subset, which we again endow with the reduced subscheme structure.

Define $\overline{\mathcal{G}}_K\text{-AdmZip} := [\overline{\mathcal{G}}_K \backslash \overline{\mathcal{G}}_K\text{-AdmZip}^\sim] \subseteq [\overline{\mathcal{G}}_K \backslash \overline{\mathcal{G}}_K\text{-Zip}^\sim] = \overline{\mathcal{G}}_K\text{-Zip}$.

Lemma 3.51. *$\overline{\mathcal{G}}_K\text{-AdmZip}^\sim$ is a closed subset of $\overline{\mathcal{G}}_K\text{-Zip}^\sim$.*

Proof: This being a purely topological question, we may freely pass to perfections, which will be convenient since Dieudonné theory is simpler over perfect rings. By “perfection” we mean the inverse perfection in the terminology of [BG18, Section 5].

Consider therefore $(\overline{\mathcal{G}}_K\text{-Zip}^\sim)^{\mathrm{perf}}$ as a sheaf on $\mathrm{Perf}_{\mathbb{F}_p}$, the fpqc site of affine perfect $\mathbb{F}_p$-schemes. Again denoting the standard Siegel lattice chain over $\mathbb{Z}_p$ by $\mathscr{V}^\bullet$ and its base change to $\mathbb{F}_p$ by $\mathcal{V}^\bullet$, we can describe the elements of $(\overline{\mathcal{G}}_K\text{-Zip}^\sim)^{\mathrm{perf}}(R) = \overline{\mathcal{G}}_K\text{-Zip}^\sim(R)$, where R is a perfect $\mathbb{F}_p$-algebra as being given by

> homomorphisms $\mathcal{V}_R^{\bullet,(p)} \xrightarrow{F^\bullet} \mathcal{V}_R^\bullet \xrightarrow{V^\bullet} \mathcal{V}_R^{\bullet,(p)}$ such that $\ker(F^\bullet) =: \mathcal{C}^{\bullet,(p)} = \mathrm{im}(V^\bullet)$ and $\mathrm{im}(F^\bullet) =: \mathcal{D}^\bullet = \ker(V^\bullet)$ and $\psi_j(F^j\_, \_) = u\sigma(\psi_j(\_, V^{-j}\_))$ for some $u \in R^\times$ and $\mathcal{C}^{\bullet,(p)}, \mathcal{D}^\bullet$ have the same rank (namely g).

To see that $\mathcal{C}^{\bullet,(p)}, \mathcal{D}^\bullet$ are direct summands of $\mathcal{V}_R^{\bullet,(p)}, \mathcal{V}_R^\bullet$ (which makes the last part of the characterization given above meaningful), one argues as in [Lau14, Lemma 2.4] (since both are finitely presented, it is enough to show flatness and to that end, one looks at the fiber dimensions).

Define a presheaf $\mathcal{X}$ on $\mathrm{Sch}_{\mathbb{Z}_p}$ in the same way but for the following changes: $\mathcal{V}^\bullet$ is replaced by $\mathscr{V}^\bullet$, and we impose the condition that both compositions $F^\bullet \circ V^\bullet$ and $V^\bullet \circ F^\bullet$ are multiplication by p, and the $\ker = \mathrm{im}$-conditions are only required to hold modulo p. We also slightly reformulate these $\ker = \mathrm{im}$-conditions: We impose the condition that the reductions $\bar{F}^\bullet, \bar{V}^\bullet$ be fiberwise of rank g over R/p. (Note that the argument that $\mathcal{C}^{\bullet,(p)}, \mathcal{D}^\bullet$ are direct summands only works over reduced rings.)

Then $\mathcal{X}$ is a separated $\mathbb{Z}_p$-scheme. To see this, we build it up from scratch as follows. $\mathrm{End}(\mathscr{V}^j)$ obviously is a $\mathbb{Z}_p$-scheme (an affine space), hence so is $\mathrm{Hom}(\mathscr{V}^{j,(p)}, \mathscr{V}^j)$ since$\mathscr{V}_j^{(p)} \cong \mathscr{V}_j$. $\mathrm{Hom}(\mathscr{V}^{\bullet,(p)}, \mathscr{V}^\bullet)$ is a locally closed subscheme of a finite product of such schemes. Homomorphisms $\mathscr{V}^{\bullet,(p)} \xrightarrow{F^\bullet} \mathscr{V}^\bullet \xrightarrow{V^\bullet} \mathscr{V}^{\bullet,(p)}$ such that both compositions are multiplication by p form a closed subscheme $\mathcal{X}'$ of $\mathrm{Hom}(\mathscr{V}^{\bullet,(p)}, \mathscr{V}^\bullet) \times \mathrm{Hom}(\mathscr{V}^\bullet, \mathscr{V}^{\bullet,(p)})$. In the special fiber $\mathcal{X}'_{\mathbb{F}_p}$ we now consider the $\ker = \mathrm{im}$-conditions and show that they define an open subscheme $\bar{\mathcal{X}}''$. Then $\mathcal{X} = \mathcal{X}' \times_{\mathcal{X}'_{\mathbb{F}_p}} \bar{\mathcal{X}}''$. Indeed, the extra conditions are that all $F^\bullet, V^\bullet$ have some non-vanishing g-minor—evidently open conditions.

The upshot is that we defined a $\mathbb{Z}_p$-scheme $\mathcal{X}$ such that $(\mathcal{X} \times_{\mathbb{Z}_p} \mathbb{F}_p)^{\mathrm{perf}} = (\overline{\mathcal{G}}_K\text{-Zip}^\sim)^{\mathrm{perf}}$ and such that we have an obvious morphism $\tilde{\mathscr{S}}_K \to \mathcal{X}$, which takes a principally polarized isogeny chain of abelian schemes to the evaluation of the Dieudonné crystal on the trivial thickening.[7]

Observe that $\mathcal{X}$ also has a natural $\mathcal{G}_K \times \mathcal{G}_K$-action: We interpret $\mathcal{G}_K$ as $\mathrm{Aut}(\mathscr{V}^\bullet)$ and the action of $(X^\bullet, Y^\bullet)$ transforms $(F^\bullet, V^\bullet)$ into $(Y^\bullet \circ F^\bullet \circ X^{\bullet,-(p)}, X^{(p)} \circ V^\bullet \circ Y^{\bullet,-1})$. The identity $(\mathcal{X} \times_{\mathbb{Z}_p} \mathbb{F}_p)^{\mathrm{perf}} = (\overline{\mathcal{G}}_K\text{-Zip}^\sim)^{\mathrm{perf}}$ is an identity of $\overline{\mathcal{G}}_K^{\mathrm{perf}} \times \overline{\mathcal{G}}_K^{\mathrm{perf}}$-varieties.

Now we claim that $\overline{\mathcal{G}}_K\text{-AdmZip}^\sim = \left(\mathcal{X}_{\mathbb{F}_p} \times_{\mathcal{X}} \overline{\mathcal{X}_{\mathbb{Q}_p}}\right)^{\mathrm{perf}}$ topologically, where $\overline{\mathcal{X}_{\mathbb{Q}_p}}$ is the flat closure of the generic fiber in $\mathcal{X}$. This of course implies $\overline{\mathcal{G}}_K\text{-AdmZip}^\sim \subseteq \overline{\mathcal{G}}_K\text{-Zip}^\sim$ being closed.

Both sets are constructible, so it suffices to check it on a very dense subset, say the $\bar{\mathbb{F}}_p$-valued points.

Using Lemmas 1.31 and 1.33, we see that $(\mathcal{X}_{\mathbb{F}_p} \times_{\mathcal{X}} \overline{\mathcal{X}_{\mathbb{Q}_p}})(\bar{\mathbb{F}}_p)$ consists precisely of those elements $\bar{x} \in \overline{\mathcal{G}}_K\text{-Zip}^\sim(\bar{\mathbb{F}}_p)$ such that there exists a finite field extension $L/\breve{\mathbb{Q}}_p$ and a point $x \in \mathcal{X}(\mathcal{O}_L)$ lifting $\bar{x}$. (We'll also say that $\bar{x}$ is *liftable* in this situation.)

Since $\mathcal{G}_K$ is flat over $\mathbb{Z}_p$, this liftability condition for $\mathcal{G}_K$ (in lieu of $\mathcal{X}$) is always satisfied. Consequently, $(\mathcal{X}_{\mathbb{F}_p} \times_{\mathcal{X}} \overline{\mathcal{X}_{\mathbb{Q}_p}})(\bar{\mathbb{F}}_p)$ is stable under the $\overline{\mathcal{G}}_K \times \overline{\mathcal{G}}_K$-action.

Also, the standard zips clearly are liftable. Thus, $(\mathcal{X}_{\mathbb{F}_p} \times_{\mathcal{X}} \overline{\mathcal{X}_{\mathbb{Q}_p}})(\bar{\mathbb{F}}_p) \supseteq \overline{\mathcal{G}}_K\text{-AdmZip}^\sim(\bar{\mathbb{F}}_p)$.

[7]This makes use of the crystalline-de Rham comparison to make a trivialization of the de Rham cohomology into a trivialization of the crystalline cohomology.

For the converse inclusion, there are injective maps from $\mathcal{X}(\mathcal{O}_L)$ to $\mathcal{X}(L)$ to $\mathcal{G}_K(L)$ such that the corresponding Schubert cell (in the local model) is indexed by the image mod $\mathcal{G}_K(\mathcal{O}_L) \times \mathcal{G}_K(\mathcal{O}_L)^{\mathrm{op}}$, cf. Proposition 3.5.[8] This proves it since we know which Schubert cells belong to the local model. □

Remark 3.52. Regarding the orbit closure relations for $\overline{\mathcal{G}}_K\text{-AdmZip}^{\sim}$, let us point out that $\overline{\mathcal{G}}_K\text{-AdmZip}^{\sim} \to \bar{M}_K^{\mathrm{loc}}$ is $\overline{\mathcal{G}}_K \times \overline{\mathcal{G}}_K$-equivariant, where the action of $\overline{\mathcal{G}}_K \times \overline{\mathcal{G}}_K$ on M_K^{loc} factors through the first projection map, and this map is a bijection on orbits. Writing $w' \preceq w$ if $(\overline{\mathcal{G}}_K \times \overline{\mathcal{G}}_K) \cdot \mathrm{Std}(w') \subseteq \overline{(\overline{\mathcal{G}}_K \times \overline{\mathcal{G}}_K) \cdot \mathrm{Std}(w)}$, it follows from these observations that $w' \leq w$ implies $w' \preceq w$. Here $\leq$ is the Bruhat order on $W_K \backslash \widetilde{W} / W_K$ as explained in [PRS13, section 4.2].

It appears reasonable to suspect that $\preceq$ and $\leq$ in fact agree.

Conjecture 3.53. The closure of $(\overline{\mathcal{G}}_K \times \overline{\mathcal{G}}_K) \cdot \widetilde{\mathrm{Std}}(w)$ is given by the disjoint union of $(\overline{\mathcal{G}}_K \times \overline{\mathcal{G}}_K) \cdot \widetilde{\mathrm{Std}}(w')$ for $w' \leq w$.

Lemma 3.54. *The map* $\overline{\mathscr{S}}_K \to \overline{\mathcal{G}}_K\text{-Zip}$ *factors through* $\overline{\mathcal{G}}_K\text{-AdmZip}$.

Proof: It is sufficient to check this on $k = \bar{\mathbb{F}}_p$-valued points.

The map $\overline{\mathscr{S}}_K(k) \to \overline{\mathcal{G}}_K\text{-Zip}(k)$ factors through $\Upsilon_K \colon \overline{\mathscr{S}}_K(k) \to \bigcup_{w \in \mathrm{KR}(K,\{\mu\})} \breve{K} w \breve{K} / \breve{K}_\sigma$ with $\breve{K} w \breve{K} / \breve{K}_\sigma \to \overline{\mathcal{G}}_K\text{-Zip}(k)$ given by sending xwy to $(\bar{y}^{-1}, \sigma(\bar{x})) \cdot \mathrm{Std}(w)$ (similar to Lemma 3.49). □

3.2.3 An explicit description of $\overline{\mathcal{G}}_K^{\mathrm{rdt}}$

In order to get a better feeling for the passage from $\overline{\mathcal{G}}_K$ to the maximal reductive quotient $\overline{\mathcal{G}}_K^{\mathrm{rdt}} = \overline{\mathcal{G}}_K / R_u \overline{\mathcal{G}}_K$ (with $R_u \overline{\mathcal{G}}_K$ being the unipotent radical of $\overline{\mathcal{G}}_K$), which is key in the definition of the EKOR stratification, we describe $\overline{\mathcal{G}}_K^{\mathrm{rdt}}$ in explicit, linear-algebraic terms in the Siegel case.

Let $(\mathcal{V}^\bullet, \mathcal{L}, \alpha_{\bullet\bullet}, \theta_\bullet, \psi_\bullet)$ be the standard Siegel lattice chain on S of type J. Assume $0 \in J$. In what follows, we sometimes use j as a shorthand for $\mathcal{V}^j$.

By a *symmetric transition map*, we mean a transition map from j' to j'', where $n \in \mathbb{Z}$, $j', j'' \in J$, $ng \geq j' \geq j'' > (n-2)g$, and $j' + j'' \in 2g\mathbb{Z}$. We will also call this the symmetric transition map of (j', n) (or of j' if n doesn't matter).

[8] Note that $\mathcal{G}_K(\mathcal{O}_L) \backslash \mathcal{G}_K(L) / \mathcal{G}_K(\mathcal{O}_L) \cong W_K \backslash \widetilde{W} / W_K$ for every strictly henselian discretely valued field L by [HR08, Prop. 8]. (And also in the construction of $\widetilde{W}$ and W_K any such field, not just $L = \breve{\mathbb{Q}}_p$, can be used.)

By a *one-sided transition map*, we mean a transition map from j' to j'', where $n \in \mathbb{Z}$, $j', j'' \in J$, $ng \geq j' \geq j'' \geq (n-1)g$. Call it right-anchored if $j' = ng$ and left-anchored if $j'' = (n-1)g$. We then also speak of the right-anchored transition map of j'' and the left-anchored transition map of j', respectively.

The kernels of the symmetric transition maps are symplectic subbundles of $\mathcal{O}_S^{2g}$ (even of the form $\mathcal{O}_S^I$, where $I \subseteq \{\pm 1, \ldots, \pm g\}$ is symmetric (i.e., $-I = I$)), and the kernels of the one-sided transition maps are totally isotropic subbundles (even of the form $\mathcal{O}_S^I$, where $I \subseteq \{1, \ldots, g\}$ or $I \subseteq \{-1, \ldots, -g\}$).

Let $\mathcal{O}_S^{I_j}$ be the kernel of the symmetric transition map of j. Then $I_j \sqcup I_{-j} = \{\pm 1, \ldots, \pm g\}$.

Every kernel of a one-sided transition map is a subbundle of a kernel of an anchored transition map inside of which it is complemented by the kernel of another one-sided transition map.

The kernel of the left-anchored transition map of j is a subbundle of the kernel of the symmetric transition map of $-j$ inside of which it is complemented by the kernel of the right-anchored transition map of $-j$. Likewise, the kernel of the right-anchored transition map of j is a subbundle of the kernel of the symmetric transition map of j inside of which it is complemented by the kernel of the left-anchored transition map of $-j$.

Now consider the standard symplectic bundle $\mathcal{O}_S^{2g}$ together with the kernels of all the symmetric transition maps and all the one-sided transition maps. So we have a symplectic bundle with a bunch of symplectic subbundles coming in complementary pairs, some of which come with a further decomposition into complementary Lagrangians, some of which come with further decompositions into complementary subbundles (of course still totally isotropic). We will also call these kernels *distinguished subspaces*.

Below we prove that $\overline{\mathcal{G}}_K^{\mathrm{rdt}}$ is the automorphism group scheme $\mathcal{A}$ of these data. Clearly, $\mathcal{A}$ is reductive; in fact it is a Levi subgroup of a parabolic of GSp_{2g}.

We have a map $\overline{\mathcal{G}}_K \to \mathcal{A}$; the image of an S-point $f^\bullet$ under $\overline{\mathcal{G}}_K \to \mathcal{A}$ on the kernel of a transition map starting at j is given by f^j. Note that $f^j = \tau \circ f^j$ on $\ker(\tau)$ for every transition map τ starting at j.

$\overline{\mathcal{G}}_K \to \mathcal{A}$ has a natural section $\mathcal{A} \to \overline{\mathcal{G}}_K$, where in the image all the f^j are the same as automorphisms of $\mathcal{O}_S^{2g}$. (This is well-defined!)

Proposition 3.55. $\mathcal{A} = \overline{\mathcal{G}}_K^{\mathrm{rdt}}$.

Proof: Let us show that $\mathcal{K} := \ker(\overline{\mathcal{G}}_K \to \mathcal{A})$ is unipotent. Consider $\overline{\mathcal{G}}_K$ as a subgroup of $\prod_{j \in J/2g\mathbb{Z}} \mathrm{GL}_{2g} \subseteq \mathrm{GL}_N$. We claim that said kernel is contained in $\prod_{j \in J/2g\mathbb{Z}} U^{(j)}$, $U^{(j)}$ being a conjugate of the standard unipotent subgroup $\begin{pmatrix} 1 & * & * & \cdots & * \\ & 1 & * & \cdots & * \\ & & \ddots & \cdots & \vdots \end{pmatrix}$ of GL_{2g}.

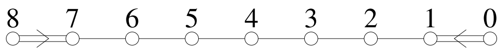
Indeed, say $f^\bullet$ is in the kernel. Then f^j acts as the identity on the kernel of the symmetric transition map of j and f^{-j} acts as the identity on the kernel of the symmetric transition map of $-j$. On the image of the symmetric transition map τ_j of j, f^{-j} agrees with $\tau_j \circ f^j$. Note that $\operatorname{im}(\tau_j) = \ker(\tau_{-j})$. So $\tau_j \circ f^j$ is the identity on $\ker(\tau_{-j})$. Hence, if $x \in \ker(\tau_{-j})$, then $x = \tau_j(x)$ and $f^j(x) \equiv x \mod \ker(\tau_j)$. Thus with respect to the decomposition $\ker(\tau_j) \oplus \ker(\tau_{-j})$, f^j is of the form $\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$.

Now we have $\overline{\mathcal{G}}_K = \mathcal{A} \ltimes \mathcal{K}$, in particular $\overline{\mathcal{G}}_K \cong \mathcal{A} \times_{\mathbb{F}_p} \mathcal{K}$ as schemes. Since both $\overline{\mathcal{G}}_K$ and $\mathcal{A}$ are reduced and connected, so is $\mathcal{K}$.

All in all, we see that $\mathcal{A}$ is indeed $\overline{\mathcal{G}}_K^{\mathrm{rdt}}$ and $\mathcal{K} = R_u\overline{\mathcal{G}}_K$ is the unipotent radical of $\overline{\mathcal{G}}_K$. $\square$

Example 3.56.

- If $J = \mathbb{Z}$, then $\overline{\mathcal{G}}_K^{\mathrm{rdt}} = \mathbb{G}_m^{g+1}$ is the standard maximal torus of GSp_{2g}.
- If $g = 2$ and $J = 2\mathbb{Z}$, then $\overline{\mathcal{G}}_K^{\mathrm{rdt}}$ is the automorphism group of the standard twisted symplectic space $\mathbb{F}_p^4$ with its standard Lagrangian decomposition, i.e., $\overline{\mathcal{G}}_K^{\mathrm{rdt}} \cong \mathrm{GL}_2 \times \mathbb{G}_m$.
- If $g = 2$ and $J/2g\mathbb{Z} = \{-1, 0, 1\}$, then $\overline{\mathcal{G}}_K^{\mathrm{rdt}}$ is the automorphism group of the standard twisted symplectic space $\mathbb{F}_p^4$ with its standard decomposition in twisted symplectic subspaces and the totally isotropic rank-1 subbundles generated by $e_{\pm 1}$, i.e., $\overline{\mathcal{G}}_K^{\mathrm{rdt}} \cong \mathrm{GL}_2 \times \mathbb{G}_m$.
- Let $g = 8$. We have the local Dynkin diagram



  where we labelled the simple affine roots as follows: $1 - 2e_{-1} + e_0$ is labelled 0, $e_{-i} - e_{-(i+1)}$ is labelled i for $1 \leq i \leq 7$, and $2e_{-8} - e_0$ is labelled 8.

  Consider $J/2g\mathbb{Z} = \{0, \pm 3, \pm 5\}$. Then the Dynkin diagram of $\overline{\mathcal{G}}_K^{\mathrm{rdt}}$ should (according to [Tit79, 3.5.1]) be the one we get by removing $0, 3, 5$ and the adjacent edges. So we expect something along the lines[9] of $\mathrm{GSp}(6) \times \mathrm{GL}(2) \times \mathrm{GL}(3)$.

  We have the following (bases of) kernels of symmetric transition maps:

$$\{\pm 1, \pm 2, \pm 3\}, \{\pm 4, \pm 5, \pm 6, \pm 7, \pm 8\}, \{\pm 1, \pm 2, \pm 3, \pm 4, \pm 5\}, \{\pm 6, \pm 7, \pm 8\},$$

[9] i.e., having the same Dynkin diagram as

and the following kernels of one-sided transition maps:

$$\{-3,-2,-1\},\{-5,-4\},\{-5,-4,-3,-2,-1\},\\ \{4,5\},\{1,2,3,4,5\},\{1,2,3\}.$$

So an element A of $\overline{\mathcal{G}}_K^{\mathrm{rdt}}$ is given by specifying linear automorphisms A_{123} of $\langle 1,2,3\rangle$ and A_{45} of $\langle 4,5\rangle$ and a symplectic similitude $A_{\pm 6,\pm 7,\pm 8}$ of $\langle \pm 6,\pm 7,\pm 8\rangle$, such that $A|_{\langle 1,2,3\rangle} = A_{123}$, $A|_{\langle 4,5\rangle} = A_{45}$, $A|_{\langle \pm 6,\pm 7,\pm 8\rangle} = A_{\pm 6,\pm 7,\pm 8}$, where $A|_{\langle -1,-2,-3\rangle}$ is uniquely determined by A_{123}, $c(A_{\pm 6,\pm 7,\pm 8})$ (c being the multiplier character) and the imposition that A be a symplectic similitude, and similarly for $A|_{\langle -4,-5\rangle}$.

If for example we consider $J/2g\mathbb{Z} = \{0,\pm 2,\pm 3,\pm 5\}$ instead, we expect something along the lines of $\mathrm{GSp}(6)\times\mathrm{GL}(2)\times\mathrm{GL}(2)$ and indeed we additionally get the subbundles

$$\{-2,-1\},\{1,2\},\{-3\},\{-5,-4,-3\},\\ \{3,4,5\},\{3\},\{3,4,5,6,7,8,-8,-7,-6,-5,-4,-3\},\{1,2,-2,-1\}.$$

So an element A of $\overline{\mathcal{G}}_K^{\mathrm{rdt}}$ is given by specifying linear automorphisms A_{12} of $\langle 1,2\rangle$ and A_{45} of $\langle 4,5\rangle$ and a symplectic similitude $A_{\pm 6,\pm 7,\pm 8}$ of $\langle \pm 6,\pm 7,\pm 8\rangle$ in a similar way to above.

3.2.4 $\overline{\mathcal{G}}_K$-EKORZip in the Siegel case

Recall that we denote the unipotent radical of $\overline{\mathcal{G}}_K$ by $R_u\overline{\mathcal{G}}_K$.

We divide out of $\overline{\mathcal{G}}_K$-$\mathrm{AdmZip}^{\sim}$ the action of the smooth normal subgroup $R_u\overline{\mathcal{G}}_K \times R_u\overline{\mathcal{G}}_K \subseteq \overline{\mathcal{G}}_K\times\overline{\mathcal{G}}_K$ and observe that $\overline{\mathcal{G}}_K\times\overline{\mathcal{G}}_K$ still acts on $[R_u\overline{\mathcal{G}}_K\times R_u\overline{\mathcal{G}}_K\backslash\overline{\mathcal{G}}_K\text{-}\mathrm{AdmZip}^{\sim}] =: \overline{\mathcal{G}}_K\text{-}\mathrm{EKORZip}^{\sim}$ (not a scheme).

We also define $\overline{\mathcal{G}}_K\text{-}\mathrm{EKORZip} := [(\Delta(\overline{\mathcal{G}}_K)\cdot(R_u\overline{\mathcal{G}}_K\times R_u\overline{\mathcal{G}}_K))\backslash\overline{\mathcal{G}}_K\text{-}\mathrm{AdmZip}^{\sim}]$.

Proposition 3.57. *We have well-defined morphisms*

$$(\overline{\mathcal{G}}_K\times\overline{\mathcal{G}}_K)/\widetilde{E}_w \to (\overline{\mathcal{G}}_K^{\mathrm{rdt}}\times\overline{\mathcal{G}}_K^{\mathrm{rdt}})/E_{\mathcal{Z}_w}, \quad (X,Y)\mapsto (X^{\mathrm{rdt}},Y^{\mathrm{rdt}}),\\ \overline{\mathcal{G}}_K/\widetilde{E}_w \to \overline{\mathcal{G}}_K^{\mathrm{rdt}}/E_{\mathcal{Z}_w}, \quad X\mapsto X^{\mathrm{rdt}},$$

and a bijectiion

$$(\overline{\mathcal{G}}_K\times\overline{\mathcal{G}}_K)/(\widetilde{E}_w\cdot(R_u\overline{\mathcal{G}}_K\times R_u\overline{\mathcal{G}}_K)) \to (\overline{\mathcal{G}}_K^{\mathrm{rdt}}\times\overline{\mathcal{G}}_K^{\mathrm{rdt}})/E_{\mathcal{Z}_w}.$$

PROOF: The first assertion follows from the definition of $E_{\mathcal{Z}_w}$ and equation (3.46). The second then follows from Lemma 3.48. $\square$

Lemma 3.58. *Assume $0 \in J$. The underlying topological spaces of the stacks in consideration are as follows:*

(1) $|[\overline{\mathcal{G}}_K \backslash \overline{M}^{\mathrm{loc}}]| = \mathrm{KR}(K, \{\mu\}) \overset{\text{def.}}{=} W_K \backslash (W_K \mathrm{Adm}(\{\mu\}) W_K)/W_K.$

(2) $|\overline{\mathcal{G}}_K\text{-EKORZip}| = \mathrm{EKOR}(K, \{\mu\}) = \mathrm{Adm}(\{\mu\})^K \cap {}^K\widetilde{W}$
$\overset{3.20}{\cong} \bigcup_{w \in \mathrm{KR}(K,\{\mu\})} \breve{K} w \breve{K}/\breve{K}_\sigma(\breve{K}_1 \times \breve{K}_1).$

PROOF: (1) is well-known as explained in Section 3.1.2.

(2): By Lemma 3.49, the $\overline{\mathcal{G}}_K \times \overline{\mathcal{G}}_K$-orbits in $\overline{\mathcal{G}}_K\text{-AdmZip}^{\sim}$ are indexed by $\mathrm{Adm}(\{\mu\})_K = \mathrm{KR}(K, \{\mu\})$.

Let us further investigate the $\overline{\mathcal{G}}_K \times \overline{\mathcal{G}}_K$-orbit of $\widetilde{\mathrm{Std}}(w)$ in $\overline{\mathcal{G}}_K\text{-EKORZip}^{\sim}$ for some fixed $w \in \mathrm{Adm}(\{\mu\})^K$.

By Proposition 3.57, its underlying topological space agrees with that of $\overline{\mathcal{G}}_K^{\mathrm{rdt}}\text{-Zip}^{\sim,\mathcal{Z}_w}$. By [SYZ19] we know that $|\overline{\mathcal{G}}_K^{\mathrm{rdt}}\text{-Zip}^{\mathcal{Z}_w}| \cong \breve{K} w \breve{K}/\breve{K}_\sigma(\breve{K}_1 \times \breve{K}_1)$, whence the lemma. $\square$

Corollary 3.59. *We have a morphism*

$$\left(\overline{\mathcal{G}}_K\text{-AdmZip}_w\right)_{\mathrm{red}} = \textit{orbit of } \mathrm{Std}(w) \to \overline{\mathcal{G}}_K^{\mathrm{rdt}}\text{-Zip}^{\mathcal{Z}_w}.$$

This defines the EKOR stratification on $\overline{\mathcal{G}}_K\text{-AdmZip}_w$. All in all, we get an EKOR stratification on $\overline{\mathcal{G}}_K\text{-AdmZip}$.

The morphism factors through $(\overline{\mathcal{G}}_K\text{-EKORZip}_w)_{\mathrm{red}}$, and $(\overline{\mathcal{G}}_K\text{-EKORZip}_w)_{\mathrm{red}} \to \overline{\mathcal{G}}_K^{\mathrm{rdt}}\text{-Zip}^{\mathcal{Z}_w}$ is an isomorphism.

Corollary 3.60. *For every point of $[\overline{\mathcal{G}}_K \backslash \overline{M}_K]$, $\overline{\mathscr{S}}_K \to \overline{\mathcal{G}}_K\text{-EKORZip}$ is smooth as a map between the associated reduced fiber of $\overline{\mathscr{S}}_K \to [\overline{\mathcal{G}}_K \backslash \overline{M}_K]$ and the associated reduced fiber of $\overline{\mathcal{G}}_K\text{-EKORZip} \to [\overline{\mathcal{G}}_K \backslash \overline{M}_K]$.*

PROOF: This follows from the preceding corollary by [SYZ19, Theorem A] (which says that the map $\overline{\mathscr{S}}_K^w \to \overline{\mathcal{G}}_K^{\mathrm{rdt}}\text{-Zip}^{\mathcal{Z}_w}$ is smooth, cf. subsection 3.1.4). $\square$

The key obstacle in going forward toward proving smoothness of $\overline{\mathscr{S}}_K \to \overline{\mathcal{G}}_K\text{-EKORZip}$ now is that we do not know whether the fibers of $\overline{\mathcal{G}}_K\text{-EKORZip} \to [\overline{\mathcal{G}}_K \backslash \overline{M}_K]$ are reduced.

Conjecture 3.61. We conjecture that the answer is affirmative. In fact, we conjecture that $\overline{\mathcal{G}}_K\text{-EKORZip} \to [\overline{\mathcal{G}}_K \backslash \overline{M}_K]$ is smooth.

Corollary 3.62. $\overline{\mathscr{S}}_K \to \overline{\mathcal{G}}_K\text{-EKORZip}$ *is surjective.*

PROOF: This follows from the description of the topological space and what is already known from [HR17, first paragraph of section 6.3]. □

We get a commutative diagram

$$\begin{tikzcd}
\tilde{\overline{\mathscr{S}}}_K \arrow[r] \arrow[d] & \overline{\mathcal{G}}_K\text{-AdmZip}^{\sim} \arrow[r] \arrow[d] & \overline{\mathcal{G}}_K\text{-EKORZip}^{\sim} \arrow[d] & \\
\overline{\mathscr{S}}_K \arrow[r] & \overline{\mathcal{G}}_K\text{-AdmZip} \arrow[r] & \overline{\mathcal{G}}_K\text{-EKORZip} \arrow[r] & {[\overline{\mathcal{G}}_K \backslash \overline{M}_K^{\mathrm{loc}}]}
\end{tikzcd}$$

Remark 3.63. Since $R_u\overline{\mathcal{G}}_K$ is smooth, $\overline{\mathcal{G}}_K\text{-AdmZip}^{\sim} \to \overline{\mathcal{G}}_K\text{-EKORZip}^{\sim}$ is smooth.

Remark 3.64. Another open question at this point is: what is the relationship between $\overline{\mathcal{G}}_K\text{-EKORZip}^{\mathrm{perf}}$ and the shtuka approach of [SYZ19, Section 4]?

Remark 3.65. It should be straightforward to generalize (taking into account the extra structure) our constructions to those (P)EL cases where the local model is the "naive" local model of Rapoport-Zink [RZ96].

3.2.5 The example of $\mathrm{GSp}(4)$

To illustrate some aspects, we look at the example $2g = 4$.

The apartment

We describe the (extended) apartment. We follow the general outline of [Lan00], in particular as far as notation is concerned.

The roots are $\pm(2e_1 - e_0), \pm(2e_2 - e_0), \pm(e_1 - e_2), \pm(e_1 + e_2 - e_0)$. The simple affine roots and the (various variants of the) Weyl group are as described in Remark 3.26. The root one-parameter subgroups[10] are given as follows:

$$u_{e_1-e_2}(x) = \begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{pmatrix}, \qquad u_{e_2-e_1}(x) = \begin{pmatrix} 1 & & & \\ x & 1 & & \\ & & 1 & \\ & & -x & 1 \end{pmatrix},$$

$$u_{2e_1-e_0}(x) = \begin{pmatrix} 1 & & & x \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \qquad u_{e_0-2e_1}(x) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ x & & & 1 \end{pmatrix},$$

$$u_{2e_2-e_0}(x) = \begin{pmatrix} 1 & & & \\ & 1 & x & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \qquad u_{e_0-2e_2}(x) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & x & 1 & \\ & & & 1 \end{pmatrix},$$

$$u_{e_1+e_2-e_0}(x) = \begin{pmatrix} 1 & & x & \\ & 1 & & x \\ & & 1 & \\ & & & 1 \end{pmatrix}, \qquad u_{e_0-e_1-e_2}(x) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ x & & 1 & \\ & x & & 1 \end{pmatrix}$$

For $a \in R$ define $w_a(x) := u_a(x)u_{-a}(-x^{-1})u_a(x)$.

Remark 3.66. $N(\mathbb{Q}_p)$ is generated by $T(\mathbb{Q}_p)$ and all $w_a(x)$ as above.

Remark 3.67. $w_a(x) = m(u_{-a}(-x^{-1}))$ in Landvogt's notation [Lan00].

We have $V_1 := X_*(T) \otimes \mathbb{R} = \{(x_1, x_2, x_{-2}, x_{-1}) \in \mathbb{R}^4 \mid x_1 + x_{-1} = x_2 + x_{-2}\}$ and

$$\nu_1 \colon T(\mathbb{Q}_p) \to V_1, \quad \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & cd_2^{-1} & \\ & & & cd_1^{-1} \end{pmatrix} \mapsto \begin{pmatrix} -v_p(d_1) \\ -v_p(d_2) \\ -v_p(cd_2^{-1}) \\ -v_p(cd_1^{-1}) \end{pmatrix}.$$

Also, $V_0 = \{v \in V_1 \mid a(v) = 0 \ \forall a \in \Phi\} = \mathbb{R}(1,1,1,1)$, $V := V_1/V_0$.

[10]The parameter being additive here; i.e., we're talking about homomorphisms $\mathbb{G}_a \to G$.

The extended apartment $A = A^{\text{ext}}$ now is an affine V_1-space together with the map $\nu_1 \colon N(\mathbb{Q}_p) \to \mathrm{Aff}(A) = \mathrm{GL}(V_1) \ltimes V_1$, whose restriction to $T(\mathbb{Q}_p)$ is given as above and (cf. Remark 3.66)

$$\nu_1(w_{2e_1-e_0}(x)) = (\begin{pmatrix} &&&1\\ &1&&\\ &&1&\\ 1&&& \end{pmatrix}, \begin{pmatrix} -v_p(x)\\ 0\\ 0\\ v_p(x) \end{pmatrix}),$$

$$\nu_1(w_{2e_2-e_0}(x)) = (\begin{pmatrix} 1&&&\\ &&1&\\ &1&&\\ &&&1 \end{pmatrix}, \begin{pmatrix} 0\\ -v_p(x)\\ v_p(x)\\ 0 \end{pmatrix}),$$

$$\nu_1(w_{e_1-e_2}(x)) = (\begin{pmatrix} &1&&\\ 1&&&\\ &&&1\\ &&1& \end{pmatrix}, \begin{pmatrix} -v_p(x)\\ v_p(x)\\ -v_p(x)\\ v_p(x) \end{pmatrix}),$$

$$\nu_1(w_{e_1+e_2-e_0}(x)) = (\begin{pmatrix} &&1&\\ &&&1\\ 1&&&\\ &1&& \end{pmatrix}, \begin{pmatrix} -v_p(x)\\ -v_p(x)\\ v_p(x)\\ v_p(x) \end{pmatrix}),$$

etc. (Recipe: Write $w_a(x)$ as a product of a diagonal matrix $\mathrm{diag}(d_1, d_2, d_{-2}, d_{-1})$ and a permutation matrix P (this need not be a factorization in $\mathrm{GSp}(4)$); then

$$\nu_1(w_a(x)) = (P, \begin{pmatrix} -v_p(d_1)\\ -v_p(d_2)\\ -v_p(d_{-2})\\ -v_p(d_{-1}) \end{pmatrix}).)$$

The reduced apartment A^{red} is the affine V-space together with $\nu \colon N(\mathbb{Q}_p) \to \mathrm{Aff}(A^{\text{red}}) = \mathrm{GL}(V) \ltimes V$ given by the same formulas.

The walls (or rather, wall conditions) are given as follows ($n \in \mathbb{Z}$):

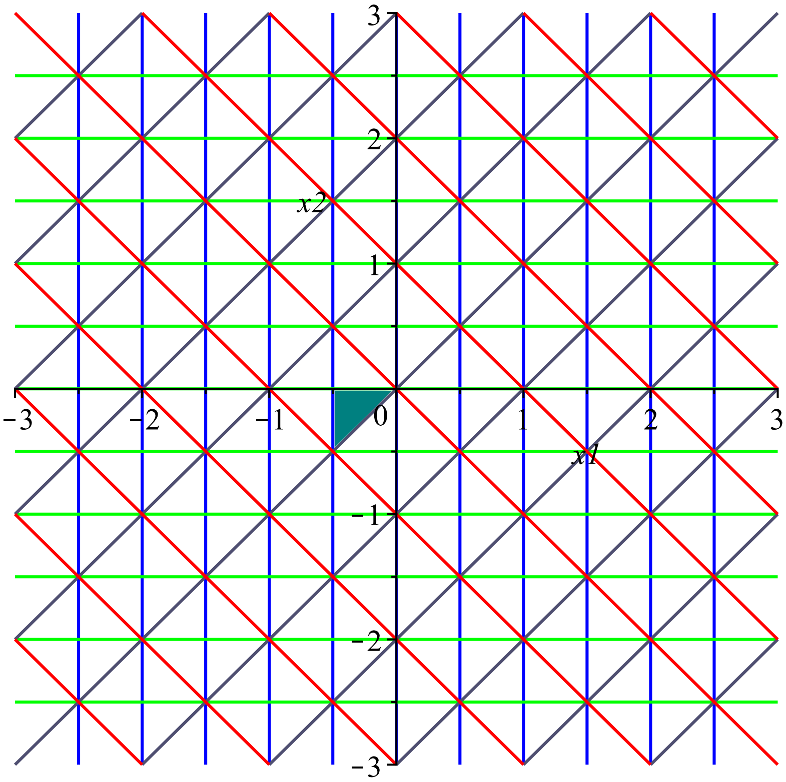
$$\begin{aligned} 2e_1 - e_0 &: n = x_0 - 2x_1, \\ 2e_2 - e_0 &: n = x_0 - 2x_2, \\ e_1 - e_2 &: n = x_2 - x_1, \\ e_1 + e_2 - e_0 &: n = x_0 - x_1 - x_2. \end{aligned}$$

Lattice chains and parahoric subgroups

By [BT84b], the extended building $\mathcal{B}(\mathrm{GL}(X), \mathbb{Q}_p)$ is in bijection with norms[11] $\alpha \colon X \to \mathbb{R} \cup \{\infty\}$. Norms in turn are in bijection with graded lattice chains (cf. Remark 1.11).

[11] Defining conditions for a norm: $\alpha(tx) = \alpha(x) + \mathrm{ord}_p(t)$, $\alpha(x+y) \geq \min(\alpha(x), \alpha(y))$, $\alpha(x) = \infty \iff x = 0$

Figure 3.1: The reduced apartment with the base alcove highlighted.



Indeed, if α is a norm, define Δ_α to be the set of its balls centered around zero and $c_\alpha(\Lambda) := \inf_{\lambda\in\Lambda} \alpha(\lambda)$. Conversely, given a graded lattice chain (Δ, c), define a norm α by $\alpha(x) := c(\Lambda)$ for the smallest $\Lambda \in \Delta$ with $x \in \Lambda$.

To go from the extended apartment of $\mathrm{GL}(X)$, an affine $\mathbb{R}^n$-space, where $n = \dim X$, to norms, fix a basis $e_1, \ldots, e_n$ of X. Then $v \in \mathbb{R}^n$ corresponds to the norm α_v with

$$\alpha_v(\sum t_i e_i) = \min_i(\mathrm{ord}_p(t_i) - v_i).$$

There are seven types of points in the extended apartment (in each case we choose one in the base alcove to represent all of its type) corresponding to the vertices, edges and interior of the base alcove:

- standard hyperspecial: $x_{\mathrm{hs}} = (0, 0, 0, 0)$
- paramodular: $x_{\mathrm{paramod}} = (-1/2, 0, 0, 1/2)$
- Klingen: $x_{\mathrm{Klingen}} = (-1/4, 0, 0, 1/4)$
- Siegel: $x_{\mathrm{Siegel}} = (-1/4, -1/4, 1/4, 1/4)$
- Iwahori: $x_{\mathrm{Iwahori}} = (-1/4, -1/8, 1/8, 1/4)$
- another hyperspecial: $x = (-1/2, -1/2, 1/2, 1/2)$
- another parahoric: $x = (-1/2, -1/4, 1/4, 1/2)$

The last two are conjugates (by the Atkin-Lehner element) of the standard hyperspecial and the Klingen parahoric, respectively (see e.g. [Rös18, p. 151]); therefore we will neglect them in the sequel.

For a set of lattices S denote by $\langle S\rangle$ the closure under homotheties, i.e., $\langle S\rangle := \{p^n s \mid n \in \mathbb{Z},\ s \in S\}$.

Then:

- $\Delta_{\mathrm{hs}} = \langle \mathbb{Z}_p^4 \rangle$
  and $c_{\mathrm{hs}}(\mathbb{Z}_p^4) = 0$.
- $\Delta_{\mathrm{paramod}} = \langle \mathbb{Z}_p^3 \oplus p\mathbb{Z}_p,\ \mathbb{Z}_p \oplus p\mathbb{Z}_p^3 \rangle$
  and $c_{\mathrm{paramod}}(\mathbb{Z}_p^3 \oplus p\mathbb{Z}_p) = -\frac{1}{2}$, $c_{\mathrm{paramod}}(\mathbb{Z}_p \oplus p\mathbb{Z}_p^3) = 0$.
- $\Delta_{\mathrm{Klingen}} = \langle \mathbb{Z}_p^4, \mathbb{Z}_p^3 \oplus p\mathbb{Z}_p, \mathbb{Z}_p \oplus p\mathbb{Z}_p^3 \rangle$
  and $c_{\mathrm{Klingen}}(\mathbb{Z}_p^4) = -1/4$, $c_{\mathrm{Klingen}}(\mathbb{Z}_p^3 \oplus p\mathbb{Z}_p) = 0$, $c_{\mathrm{Klingen}}(\mathbb{Z}_p \oplus p\mathbb{Z}_p^3) = 1/4$.

- $\Delta_{\mathrm{Siegel}} = \langle \mathbb{Z}_p^4, \mathbb{Z}_p^2 \oplus p\mathbb{Z}_p^2 \rangle$
  and $c_{\mathrm{Siegel}}(\mathbb{Z}_p^4) = -1/4$, $c_{\mathrm{Siegel}}(\mathbb{Z}_p^2 \oplus p\mathbb{Z}_p^2) = 1/4$.
- $\Delta_{\mathrm{Iwahori}} = \langle \mathbb{Z}_p^4, \mathbb{Z}_p^3 \oplus p\mathbb{Z}_p, \mathbb{Z}_p^2 \oplus p\mathbb{Z}_p^2, \mathbb{Z}_p \oplus p\mathbb{Z}_p^3 \rangle$
  and $c_{\mathrm{Iwahori}}(\mathbb{Z}_p^4) = -1/4$, $c_{\mathrm{Iwahori}}(\mathbb{Z}_p^3 \oplus p\mathbb{Z}_p) = -1/8$, $c_{\mathrm{Iwahori}}(\mathbb{Z}_p^2 \oplus p\mathbb{Z}_p^2) = 1/8$, $c_{\mathrm{Iwahori}}(\mathbb{Z}_p \oplus p\mathbb{Z}_p^3) = 1/4$.

The associated parahoric subgroups are

- hyperspecial: $\mathrm{GSp}_4(\mathbb{Z}_p)$
- paramodular: $\mathrm{GSp}_4(\mathbb{Q}_p) \cap \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & p^{-1}\mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & p\mathbb{Z}_p & p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$
- Klingen: $\mathrm{GSp}_4(\mathbb{Z}_p) \cap \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & p\mathbb{Z}_p & p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$
- Siegel: $\mathrm{GSp}_4(\mathbb{Z}_p) \cap \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$
- Iwahori: $\mathrm{GSp}_4(\mathbb{Z}_p) \cap \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & p\mathbb{Z}_p & p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$

Remark 3.68. Dualizing with respect to the symplectic form, we have

$$\begin{aligned} (\mathbb{Z}_p^4)^\vee &= \mathbb{Z}_p^4, & (\mathbb{Z}_p \oplus p\mathbb{Z}_p^3)^\vee &= p^{-1}(\mathbb{Z}_p^3 \oplus p\mathbb{Z}_p), \\ (\mathbb{Z}_p^3 \oplus p\mathbb{Z}_p)^\vee &= p^{-1}(\mathbb{Z}_p \oplus p\mathbb{Z}_p^3), & (\mathbb{Z}_p^2 \oplus p\mathbb{Z}_p^2)^\vee &= p^{-1}(\mathbb{Z}_p^2 \oplus p\mathbb{Z}_p^2). \end{aligned}$$

Admissible set

We compute the admissible set in the way outlined in Remark 3.29. The cocharacter μ is $(1, 1, 0, 0)$.

We obtain

$$\begin{aligned}
\operatorname{Adm}(\{\mu\}) = \{&\Bigl(\mathrm{id}, \left(\begin{smallmatrix}1\\1\\0\\0\end{smallmatrix}\right)\Bigr), \quad \Bigl((2 \quad -2), \left(\begin{smallmatrix}1\\0\\1\\0\end{smallmatrix}\right)\Bigr), \quad \Bigl(\mathrm{id}, \left(\begin{smallmatrix}1\\0\\1\\0\end{smallmatrix}\right)\Bigr),\\
&\Bigl((1 \quad -1), \left(\begin{smallmatrix}0\\1\\0\\1\end{smallmatrix}\right)\Bigr), \quad \Bigl((1 \quad 2 \quad -1 \quad -2), \left(\begin{smallmatrix}0\\1\\0\\1\end{smallmatrix}\right)\Bigr),\\
&\Bigl((1 \quad 2)(-2 \quad -1), \left(\begin{smallmatrix}0\\1\\0\\1\end{smallmatrix}\right)\Bigr), \quad \Bigl(\mathrm{id}, \left(\begin{smallmatrix}0\\1\\0\\1\end{smallmatrix}\right)\Bigr),\\
&\Bigl((1 \quad -1)(2 \quad -2), \left(\begin{smallmatrix}0\\0\\1\\1\end{smallmatrix}\right)\Bigr), \quad \Bigl((1 \quad -1), \left(\begin{smallmatrix}0\\0\\1\\1\end{smallmatrix}\right)\Bigr),\\
&\Bigl((1 \quad -2)(2 \quad -1), \left(\begin{smallmatrix}0\\0\\1\\1\end{smallmatrix}\right)\Bigr), \quad \Bigl((1 \quad -2 \quad -1 \quad 2), \left(\begin{smallmatrix}0\\0\\1\\1\end{smallmatrix}\right)\Bigr),\\
&\Bigl((2 \quad -2), \left(\begin{smallmatrix}0\\0\\1\\1\end{smallmatrix}\right)\Bigr), \quad \Bigl(\mathrm{id}, \left(\begin{smallmatrix}0\\0\\1\\1\end{smallmatrix}\right)\Bigr)\},
\end{aligned}$$

or, in terms of Frobenii (cf. Construction 3.43)

$$\begin{aligned}
\Bigl\{&\begin{pmatrix}p&0&0&0\\0&p&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}p&0&0&0\\0&0&1&0\\0&p&0&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}p&0&0&0\\0&1&0&0\\0&0&p&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0&0&0&1\\0&p&0&0\\0&0&1&0\\p&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&1&0\\p&0&0&0\\0&0&0&1\\0&p&0&0\end{pmatrix},\\
&\begin{pmatrix}0&1&0&0\\p&0&0&0\\0&0&0&1\\0&0&p&0\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&p&0&0\\0&0&1&0\\0&0&0&p\end{pmatrix}, \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&p&0&0\\p&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&1\\0&1&0&0\\0&0&p&0\\p&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&1&0\\0&0&0&1\\p&0&0&0\\0&p&0&0\end{pmatrix},\\
&\begin{pmatrix}0&1&0&0\\0&0&0&1\\p&0&0&0\\0&0&p&0\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&0&1&0\\0&p&0&0\\0&0&0&p\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&p&0\\0&0&0&p\end{pmatrix}\Bigr\}.
\end{aligned}$$

Siegel level

From now on, we consider the Siegel level structure. Denote the Siegel parahoric by K and the standard hyperspecial subgroup by H. Here W_K is generated by $(-1 \quad -2)(1 \quad 2)$, while W_H is generated by W_K and $(2 \quad -2)$.

Recalling Remark 3.31 (2), we note that one has a natural morphism $\overline{\mathcal{G}}_K\text{-Zip} \to \overline{\mathcal{G}}_H\text{-Zip} \times \overline{\mathcal{G}}_H\text{-Zip}$.

We have

$$\begin{aligned}
\mathrm{KR}(K,\{\mu\}) = \Big\{&(\mathrm{id}, \left(\begin{smallmatrix}1\\1\\0\\0\end{smallmatrix}\right)), ((1\quad -2)(2\quad -1), \left(\begin{smallmatrix}0\\0\\1\\1\end{smallmatrix}\right)), ((1\quad 2\quad -1\quad -2), \left(\begin{smallmatrix}0\\1\\0\\1\end{smallmatrix}\right)),\\
&(\mathrm{id}, \left(\begin{smallmatrix}0\\0\\1\\1\end{smallmatrix}\right)), ((1\quad -2\quad -1\quad 2), \left(\begin{smallmatrix}0\\0\\1\\1\end{smallmatrix}\right)), ((1\quad 2)(-2\quad -1), \left(\begin{smallmatrix}0\\1\\0\\1\end{smallmatrix}\right))\Big\},\\
\mathrm{EKOR}(K,\{\mu\}) = \mathrm{KR}(K,\{\mu\}) \cup \Big\{&(\mathrm{id}, \left(\begin{smallmatrix}0\\1\\0\\1\end{smallmatrix}\right)), \quad ((2\quad -2), \left(\begin{smallmatrix}0\\0\\1\\1\end{smallmatrix}\right)), \quad ((1\quad -1), \left(\begin{smallmatrix}0\\1\\0\\1\end{smallmatrix}\right))\Big\}.
\end{aligned}$$

In the following table, w^j is the isomorphism type of the $\overline{\mathcal{G}}_H$-zip at position j. For $\mathcal{C}^\bullet, \mathcal{D}^\bullet$ we give (indices of) basis vectors. "$\leftarrow$" means "same as in the column adjacent to the left". $\alpha_0 \colon \bar{\mathbb{F}}_p^4 \to \bar{\mathbb{F}}_p^4$ is the projection onto the plane spanned by the $1, 2$-coordinates, α_2 the projection onto the plane spanned by the $-2, -1$-coordinates. By $\alpha_{j,\mathcal{C}^\bullet/\mathcal{D}^\bullet}$ we denote the induced maps on $\mathcal{V}^\bullet/\mathcal{C}^\bullet \oplus \mathcal{C}^\bullet$ and $\mathcal{D}^\bullet \oplus \mathcal{V}^\bullet/\mathcal{D}^\bullet$, respectively. Each $\mathcal{C}^j \subseteq \mathcal{V}^j$ has a canonical complement in terms of standard basis vectors. Importantly however, we will not always have a complementary *chain* of linear subspaces. In any event, below we say what the $\alpha_{j,\mathcal{C}^\bullet/\mathcal{D}^\bullet}$ are the projection onto if interpreted as described. For instance, the projection onto $\emptyset$ is the zero map. So in that case $\mathcal{V}^\bullet/\mathcal{C}^\bullet \oplus \mathcal{C}^\bullet$ (or $\mathcal{D}^\bullet \oplus \mathcal{V}^\bullet/\mathcal{D}^\bullet$) is a chain of vector spaces with zero transition maps.

w	KR-type	$\mathcal{C}^0$	$\mathcal{D}^0$	$\mathcal{C}^2$	$\mathcal{D}^2$	w^0	w^2	$\alpha_{2,\mathcal{C}^\bullet}$	$\alpha_{0,\mathcal{C}^\bullet}$	$\alpha_{2,\mathcal{D}^\bullet}$	$\alpha_{0,\mathcal{D}^\bullet}$
$(\mathrm{id}, \begin{pmatrix} 1\\1\\0\\0 \end{pmatrix})$	$\leftarrow$	$\{1,2\}$	$\{1,2\}$	$\{-2,-1\}$	$\{-2,-1\}$	$(-2\ \ 1)(-1\ \ 2)$	$\leftarrow$	$\{1,2\}$	$\{-2,-1\}$	$\{1,2\}$	$\{-2,-1\}$
$((1\ \ -2)(2\ \ -1), \begin{pmatrix} 0\\0\\1\\1 \end{pmatrix})$	$\leftarrow$	$\{1,2\}$	$\{-2,-1\}$	$\{1,2\}$	$\{-2,-1\}$	id	$\leftarrow$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$((1\ \ 2\ \ -1\ \ -2), \begin{pmatrix} 0\\1\\0\\1 \end{pmatrix})$	$\leftarrow$	$\{1,2\}$	$\{-2,1\}$	$\{-2,1\}$	$\{-2,-1\}$	$(-2\ \ 2)$	$\leftarrow$	$\{1\}$	$\{-1\}$	$\{2\}$	$\{-2\}$
$(\mathrm{id}, \begin{pmatrix} 0\\0\\1\\1 \end{pmatrix})$	$\leftarrow$	$\{-2,-1\}$	$\{-2,-1\}$	$\{1,2\}$	$\{1,2\}$	$(-2\ \ 1)(-1\ \ 2)$	$\leftarrow$	$\{1,2\}$	$\{-2,-1\}$	$\{1,2\}$	$\{-2,-1\}$
$((1\ \ -2\ \ -1\ \ 2), \begin{pmatrix} 0\\0\\1\\1 \end{pmatrix})$	$\leftarrow$	$\{-2,1\}$	$\{-2,-1\}$	$\{1,2\}$	$\{-2,1\}$	$(-2\ \ 2)$	$\leftarrow$	$\{2\}$	$\{-2\}$	$\{1\}$	$\{-1\}$
$((1\ \ 2)(-2\ \ -1), \begin{pmatrix} 0\\1\\0\\1 \end{pmatrix})$	$\leftarrow$	$\{-2,1\}$	$\{-2,1\}$	$\{-2,1\}$	$\{-2,1\}$	id	$\leftarrow$	$\{1,2\}$	$\{-2,-1\}$	$\{1,2\}$	$\{-2,-1\}$
$(\mathrm{id}, \begin{pmatrix} 0\\1\\0\\1 \end{pmatrix})$	$((1\ \ 2)(-2\ \ -1), \begin{pmatrix} 0\\1\\0\\1 \end{pmatrix})$	$\{-1,2\}$	$\{-1,2\}$	$\{-2,1\}$	$\{-2,1\}$	$(-2\ \ 1)(-1\ \ 2)$	$\leftarrow$	$\{1,2\}$	$\{-2,-1\}$	$\{1,2\}$	$\{-2,-1\}$
$((2\ \ -2), \begin{pmatrix} 0\\0\\1\\1 \end{pmatrix})$	$((1\ \ -2\ \ -1\ \ 2), \begin{pmatrix} 0\\0\\1\\1 \end{pmatrix})$	$\{-1,2\}$	$\{-2,-1\}$	$\{1,2\}$	$\{-2,1\}$	$(-2\ \ -1\ \ 2\ \ 1)$	$\leftarrow$	$\{1\}$	$\{-1\}$	$\{1\}$	$\{-1\}$
$((1\ \ -1), \begin{pmatrix} 0\\1\\0\\1 \end{pmatrix})$	$((1\ \ 2\ \ -1\ \ -2), \begin{pmatrix} 0\\1\\0\\1 \end{pmatrix})$	$\{1,2\}$	$\{-1,2\}$	$\{-2,1\}$	$\{-2,-1\}$	$(-2\ \ -1\ \ 2\ \ 1)$	$\leftarrow$	$\{2\}$	$\{-2\}$	$\{2\}$	$\{-2\}$

Observations 3.69.

- We always have $w^0 = w^2$. This is explained by the fact that the Ekedahl-Oort stratification in this case agrees with the Newton stratification (and isogenous abelian varieties by definition lie in the same Newton stratum).
- Consider the Kottwitz-Rapoport strata containing more than one EKOR stratum (i.e., containing two EKOR strata). Then we can distinguish among the EKOR strata by looking at the Ekedahl-Oort stratum. In other words, the EKOR stratification is in this case the coarsest common refinement of the Kottwitz-Rapoport and Ekedahl-Oort stratifications.

Bibliography

[Ahs11] Tobias Ahsendorf. “$\mathcal{O}$-displays and π-divisible formal $\mathcal{O}$-modules”. PhD thesis. Universität Bielefeld, 2011. URL: `http://nbn-resolving.de/urn:nbn:de:hbz:361-24713520`.

[AT08] Alexander Arhangel’skii and Mikhail Tkachenko. *Topological groups and related structures*. Hackensack, NJ: World Scientific; Paris: Atlantis Press, 2008. ISBN: 978-90-78677-06-2/hbk.

[BB05] Anders Björner and Francesco Brenti. *Combinatorics of Coxeter groups*. Vol. 231. Graduate Texts in Mathematics. Springer, New York, 2005, pp. xiv+363. ISBN: 978-3540-442387; 3-540-44238-3.

[BBM82] Pierre Berthelot, Lawrence Breen, and William Messing. *Théorie de Dieudonné cristalline. II*. Vol. 930. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1982, pp. x+261. ISBN: 3-540-11556-0. DOI: `10.1007/BFb0093025`.

[BG18] Alessandra Bertapelle and Cristian D. González-Avilés. “On the perfection of schemes”. In: *Expo. Math.* 36.2 (2018), pp. 197–220. ISSN: 0723-0869.

[BM90] Pierre Berthelot and William Messing. “Théorie de Dieudonné cristalline. III. Théorèmes d’équivalence et de pleine fidélité”. In: *The Grothendieck Festschrift, Vol. I*. Vol. 86. Progr. Math. Birkhäuser Boston, Boston, MA, 1990, pp. 173–247.

[Bor69] Armand Borel. *Introduction aux groupes arithmétiques*. Publications de l’Institut de Mathématique de l’Université de Strasbourg, XV. Actualités Scientifiques et Industrielles, No. 1341. Hermann, Paris, 1969.

[BT84a] François Bruhat and Jacques Tits. “Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d’une donnée radicielle valuée.” French. In: *Publ. Math., Inst. Hautes Étud. Sci.* 60 (1984), pp. 1–194. ISSN: 0073-8301; 1618-1913/e.

[BT84b] François Bruhat and Jacques Tits. “Schémas en groupes et immeubles des groupes classiques sur un corps local”. In: *Bull. Soc. Math. Fr.* 112 (1984), pp. 259–301. ISSN: 0037-9484; 2102-622X/e. DOI: `10.24033/bsmf.2006`.

[Car93] Roger W. Carter. *Finite groups of Lie type*. Wiley Classics Library. Conjugacy classes and complex characters, Reprint of the 1985 original, A Wiley-Interscience Publication. John Wiley & Sons, Ltd., Chichester, 1993, pp. xii+544. ISBN: 0-471-94109-3.

[CS17] Ana Caraiani and Peter Scholze. "On the generic part of the cohomology of compact unitary Shimura varieties." In: *Ann. Math. (2)* 186.3 (2017), pp. 649–766. ISSN: 0003-486X; 1939-8980/e. DOI: 10.4007/annals.2017.186.3.1.

[Dan36] D. van Dantzig. "Zur topologischen Algebra. III: Brouwersche und Cantorsche Gruppen". In: *Compos. Math.* 3 (1936), pp. 408–426. ISSN: 0010-437X; 1570-5846/e.

[Del11] Pierre Deligne. *Letter to Kisin*. http://people.math.binghamton.edu/adrian/Letter_Deligne.pdf. Aug. 2011.

[Del71] Pierre Deligne. "Travaux de Shimura". In: *Séminaire Bourbaki, 23ème année (1970/71), Exp. No. 389*. Vol. 244. Lecture Notes in Math. Springer, Berlin, 1971, pp. 123–165.

[Del82] Pierre Deligne. *Hodge cycles on abelian varieties. (Notes by J. S. Milne)*. Hodge cycles, motives, and Shimura varieties, Lect. Notes Math. 900, 9-100 (1982). 1982.

[DG80] Michel Demazure and Peter Gabriel. *Introduction to algebraic geometry and algebraic groups*. Vol. 39. North-Holland Mathematics Studies. Translated from the French by J. Bell. North-Holland Publishing Co., Amsterdam-New York, 1980, pp. xiv+357. ISBN: 0-444-85443-6.

[EGA2] A. Grothendieck. "Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes". In: *Inst. Hautes Études Sci. Publ. Math.* 8 (1961). ISSN: 0073-8301. URL: http://www.numdam.org/item?id=PMIHES_1961__8__222_0.

[EGA4] Alexander Grothendieck. *Éléments de géométrie algébrique IV: Étude locale des schémas et des morphismes de schémas*. Publications mathématiques de l'I.H.É.S., 1964–1967.

[Gör03] Ulrich Görtz. "On the flatness of local models for the symplectic group". In: *Adv. Math.* 176.1 (2003), pp. 89–115. ISSN: 0001-8708.

[Gro74] Alexandre Grothendieck. *Groupes de Barsotti-Tate et cristaux de Dieudonné*. Séminaire de Mathématiques Supérieures, No. 45 (Été, 1970). Les Presses de l'Université de Montréal, Montreal, Que., 1974.

[GW10] Ulrich Görtz and Torsten Wedhorn. *Algebraic geometry I*. Advanced Lectures in Mathematics. Schemes with examples and exercises. Vieweg + Teubner, Wiesbaden, 2010. ISBN: 978-3-8348-0676-5. DOI: 10.1007/978-3-8348-9722-0.

[Hai05] Thomas J. Haines. "Introduction to Shimura varieties with bad reduction of parahoric type". In: *Harmonic analysis, the trace formula, and Shimura varieties. Proceedings of the Clay Mathematics Institute 2003 summer school, Toronto, Canada, June 2–27, 2003*. Providence, RI: American Mathematical Society (AMS), 2005, pp. 583–642. ISBN: 0-8218-3844-X/pbk.

[Ham17] Paul Hamacher. "The almost product structure of Newton strata in the deformation space of a Barsotti-Tate group with crystalline Tate tensors". In: *Math. Z.* 287.3-4 (2017), pp. 1255–1277. ISSN: 0025-5874. DOI: 10.1007/s00209-017-1867-2.

[Haz78] Michiel Hazewinkel. *Formal groups and applications*. Pure and Applied Mathematics, 78. New York-San Francisco-London: Academic Press. XXII, 573 p. 1978.

[HK17] P. Hamacher and W. Kim. *l-adic étale cohomology of Shimura varieties of Hodge type with non-trivial coefficients*. Preprint. Nov. 2017. arXiv: 1711.07123v1 [math.NT]. Published version: "l-adic étale cohomology of Shimura varieties of Hodge type with non-trivial coefficients". In: *Math. Ann.* 375.3-4 (2019), pp. 973–1044. ISSN: 0025-5831; 1432-1807/e.

[HR08] Thomas J. Haines and Michael Rapoport. "Appendix: On parahoric subgroups". In: *Advances in Mathematics* 219.1 (2008), pp. 188–198. ISSN: 0001-8708. DOI: https://doi.org/10.1016/j.aim.2008.04.020.

[HR17] X. He and M. Rapoport. "Stratifications in the reduction of Shimura varieties." In: *Manuscr. Math.* 152.3-4 (2017), pp. 317–343. ISSN: 0025-2611; 1432-1785/e. DOI: 10.1007/s00229-016-0863-x.

[Ill85] Luc Illusie. "Déformations de groupes de Barsotti-Tate (d'après A. Grothendieck)". In: *Astérisque* 127 (1985). Seminar on arithmetic bundles: the Mordell conjecture (Paris, 1983/84), pp. 151–198. ISSN: 0303-1179.

[Jon95] A. J. de Jong. "Crystalline Dieudonné module theory via formal and rigid geometry". In: *Inst. Hautes Études Sci. Publ. Math.* 82 (1995), pp. 5–96. ISSN: 0073-8301. URL: http://www.numdam.org/item?id=PMIHES_1995__82__5_0.

[Kim19] Wansu Kim. "On central leaves of Hodge-type Shimura varieties with parahoric level structure". In: *Math. Z.* 291.1-2 (2019), pp. 329–363. ISSN: 0025-5874; 1432-1823/e.

[Kis10] Mark Kisin. "Integral models for Shimura varieties of abelian type". In: *J. Amer. Math. Soc.* 23.4 (2010), pp. 967–1012. ISSN: 0894-0347. DOI: 10.1090/S0894-0347-10-00667-3.

[Kis17] Mark Kisin. "Mod p points on Shimura varieties of abelian type". In: *J. Am. Math. Soc.* 30.3 (2017), pp. 819–914. ISSN: 0894-0347; 1088-6834/e.

[Kot92] Robert E. Kottwitz. “Points on some Shimura varieties over finite fields”. In: *J. Amer. Math. Soc.* 5.2 (1992), pp. 373–444. ISSN: 0894-0347. DOI: 10.2307/2152772.

[Kot97] Robert E. Kottwitz. “Isocrystals with additional structure. II”. In: *Compositio Math.* 109.3 (1997), pp. 255–339. ISSN: 0010-437X. DOI: 10.1023/A:1000102604688.

[KP15] M. Kisin and G. Pappas. *Integral models of Shimura varieties with parahoric level structure*. Preprint. Dec. 2015. arXiv: 1512.01149v2 [math.AG]. Published version: “Integral models of Shimura varieties with parahoric level structure”. In: *Publ. Math., Inst. Hautes Étud. Sci.* 128 (2018), pp. 121–218. ISSN: 0073-8301; 1618-1913/e.

[KR00] R. Kottwitz and M. Rapoport. “Minuscule alcoves for GL_n and GSp_{2n}”. In: *Manuscripta Math.* 102.4 (2000), pp. 403–428. ISSN: 0025-2611. DOI: 10.1007/s002290070034.

[Lan00] Erasmus Landvogt. “Some functorial properties of the Bruhat-Tits building”. In: *J. Reine Angew. Math.* 518 (2000), pp. 213–241. ISSN: 0075-4102; 1435-5345/e. DOI: 10.1515/crll.2000.006.

[Lan96] Erasmus Landvogt. *A compactification of the Bruhat-Tits building*. Vol. 1619. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1996, pp. viii+152. ISBN: 3-540-60427-8. DOI: 10.1007/BFb0094594.

[Lau13] Eike Lau. “Smoothness of the truncated display functor.” In: *J. Am. Math. Soc.* 26.1 (2013), pp. 129–165. ISSN: 0894-0347; 1088-6834/e.

[Lau14] Eike Lau. “Relations between Dieudonné displays and crystalline Dieudonné theory”. In: *Algebra Number Theory* 8.9 (2014), pp. 2201–2262. ISSN: 1937-0652. DOI: 10.2140/ant.2014.8.2201.

[Man04] Elena Mantovan. “On certain unitary group Shimura varieties”. In: *Astérisque* 291 (2004). Variétés de Shimura, espaces de Rapoport-Zink et correspondances de Langlands locales, pp. 201–331. ISSN: 0303-1179.

[Man05] Elena Mantovan. “On the cohomology of certain PEL-type Shimura varieties”. In: *Duke Math. J.* 129.3 (2005), pp. 573–610. ISSN: 0012-7094. DOI: 10.1215/S0012-7094-05-12935-0.

[Mes72] William Messing. *The crystals associated to Barsotti-Tate groups: with applications to abelian schemes*. Lecture Notes in Mathematics, Vol. 264. Springer-Verlag, Berlin-New York, 1972.

[Mil05] J. S. Milne. "Introduction to Shimura varieties." In: *Harmonic analysis, the trace formula, and Shimura varieties. Proceedings of the Clay Mathematics Institute 2003 summer school, Toronto, Canada, June 2–27, 2003*. Providence, RI: American Mathematical Society (AMS), 2005, pp. 265–378. ISBN: 0-8218-3844-X/pbk.

[Mor93] Lawrence Morris. "Tamely ramified intertwining algebras". In: *Invent. Math.* 114.1 (1993), pp. 1–54. ISSN: 0020-9910. DOI: `10.1007/BF01232662`.

[MW04] Ben Moonen and Torsten Wedhorn. "Discrete invariants of varieties in positive characteristic". In: *Int. Math. Res. Not.* 72 (2004), pp. 3855–3903. ISSN: 1073-7928. DOI: `10.1155/S1073792804141263`.

[Oor04] Frans Oort. "Foliations in moduli spaces of abelian varieties". In: *J. Amer. Math. Soc.* 17.2 (2004), pp. 267–296. ISSN: 0894-0347. DOI: `10.1090/S0894-0347-04-00449-7`.

[Oor09] Frans Oort. "Foliations in moduli spaces of abelian varieties and dimension of leaves". In: *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II*. Vol. 270. Progr. Math. Birkhäuser Boston, Inc., Boston, MA, 2009, pp. 465–501. DOI: `10.1007/978-0-8176-4747-6_15`.

[OZ02] Frans Oort and Thomas Zink. "Families of p-divisible groups with constant Newton polygon". In: *Doc. Math.* 7 (2002), pp. 183–201. ISSN: 1431-0635.

[Pin90] Richard Pink. *Arithmetical compactification of mixed Shimura varieties*. Vol. 209. Bonner Mathematische Schriften [Bonn Mathematical Publications]. Universität Bonn, Mathematisches Institut, 1990.

[PRS13] Georgios Pappas, Michael Rapoport, and Brian Smithling. "Local models of Shimura varieties, I. Geometry and combinatorics". In: Adv. Lect. Math. (ALM) 26 (2013), pp. 135–217.

[PWZ11] Richard Pink, Torsten Wedhorn, and Paul Ziegler. "Algebraic zip data". In: *Doc. Math.* 16 (2011), pp. 253–300. ISSN: 1431-0635; 1431-0643/e.

[PWZ15] Richard Pink, Torsten Wedhorn, and Paul Ziegler. "F-zips with additional structure". In: *Pac. J. Math.* 274.1 (2015), pp. 183–236. ISSN: 0030-8730. DOI: `10.2140/pjm.2015.274.183`.

[PZ13] Georgios Pappas and Xinwen Zhu. "Local models of Shimura varieties and a conjecture of Kottwitz". In: *Invent. Math.* 194.1 (2013), pp. 147–254. ISSN: 0020-9910; 1432-1297/e. DOI: `10.1007/s00222-012-0442-z`.

[Rap05] Michael Rapoport. "A guide to the reduction modulo p of Shimura varieties." In: *Formes automorphes (I). Actes du Semestre du Centre Émile Borel, Paris, France, 17 février au 11 juillet 2000*. Paris: Société Mathématique de France, 2005, pp. 271–318. ISBN: 2-85629-172-4/pbk.

[Rös18] Mirko Rösner. “Parahoric restriction for $\mathrm{GSp}(4)$.” In: *Algebr. Represent. Theory* 21.1 (2018), pp. 145–161. ISSN: 1386-923X; 1572-9079/e.

[RV14] Michael Rapoport and Eva Viehmann. “Towards a theory of local Shimura varieties”. In: *Münster J. Math.* 7.1 (2014), pp. 273–326. ISSN: 1867-5778; 1867-5786/e.

[RZ96] M. Rapoport and Th. Zink. *Period spaces for p-divisible groups*. Vol. 141. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 1996. DOI: 10.1515/9781400882601.

[Ser97] Jean-Pierre Serre. *Galois cohomology*. Translated from the French by Patrick Ion. Springer-Verlag, Berlin, 1997, pp. x+210. ISBN: 3-540-61990-9. DOI: 10.1007/978-3-642-59141-9.

[SGA3] *Schémas en groupes*. Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vols. 151–153. Springer-Verlag, Berlin-New York, 1970.

[Stacks] The Stacks Project Authors. *Stacks Project*. http://stacks.math.columbia.edu. 2020.

[SW13] Peter Scholze and Jared Weinstein. “Moduli of p-divisible groups”. In: *Camb. J. Math.* 1.2 (2013), pp. 145–237. ISSN: 2168-0930; 2168-0949/e. DOI: 10.4310/CJM.2013.v1.n2.a1.

[SYZ19] Xu Shen, Chia-Fu Yu, and Chao Zhang. *EKOR strata for Shimura varieties with parahoric level structure*. Preprint. Oct. 2019. arXiv: 1910.07785v1 [math.AG].

[Tit79] J. Tits. “Reductive groups over local fields”. In: *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*. Proc. Sympos. Pure Math., XXXIII. Amer. Math. Soc., Providence, R.I., 1979, pp. 29–69.

[Vas08] Adrian Vasiu. “Level m stratifications of versal deformations of p-divisible groups”. In: *J. Algebraic Geom.* 17.4 (2008), pp. 599–641. ISSN: 1056-3911. DOI: 10.1090/S1056-3911-08-00495-5.

[VW13] Eva Viehmann and Torsten Wedhorn. “Ekedahl-Oort and Newton strata for Shimura varieties of PEL type”. In: *Math. Ann.* 356.4 (2013), pp. 1493–1550. ISSN: 0025-5831. DOI: 10.1007/s00208-012-0892-z.

[Wed99] Torsten Wedhorn. “Ordinariness in good reductions of Shimura varieties of PEL-type”. In: *Ann. Sci. École Norm. Sup. (4)* 32.5 (1999), pp. 575–618. ISSN: 0012-9593. DOI: 10.1016/S0012-9593(01)80001-X.

[Wor13] D. Wortmann. *The μ-ordinary locus for Shimura varieties of Hodge type*. Preprint. Oct. 2013. arXiv: 1310.6444v1 [math.AG].

[Yu08] Chia-Fu Yu. “Irreducibility and p-adic monodromies on the Siegel moduli spaces”. In: *Adv. Math.* 218.4 (2008), pp. 1253–1285. ISSN: 0001-8708.

[Zha15] C. Zhang. *Stratifications and foliations for good reductions of Shimura varieties of Hodge type*. Preprint. Dec. 2015. arXiv: `1512.08102v1 [math.AG]`.

[Zha18] Chao Zhang. "Ekedahl-Oort strata for good reductions of Shimura varieties of Hodge type". In: *Canad. J. Math.* 70.2 (2018), pp. 451–480. ISSN: 0008-414X. DOI: `10.4153/CJM-2017-020-5`.

[Zho18] R. Zhou. *Mod-p isogeny classes on Shimura varieties with parahoric level structure*. Preprint. Feb. 2018. arXiv: `1707.09685v2 [math.NT]`.

[Zhu17] Xinwen Zhu. "Affine Grassmannians and the geometric Satake in mixed characteristic." In: *Ann. Math. (2)* 185.2 (2017), pp. 403–492. ISSN: 0003-486X; 1939-8980/e.

[Zin01a] Thomas Zink. "A Dieudonné theory for p-divisible groups." In: *Class field theory – its centenary and prospect. Proceedings of the 7th MSJ International Research Institute of the Mathematical Society of Japan, Tokyo, Japan, June 3–12, 1998*. Tokyo: Mathematical Society of Japan, 2001, pp. 139–160. ISBN: 4-931469-11-6/hbk.

[Zin01b] Thomas Zink. "On the slope filtration". In: *Duke Math. J.* 109.1 (2001), pp. 79–95. ISSN: 0012-7094. DOI: `10.1215/S0012-7094-01-10913-7`.

[Zin02] Thomas Zink. "The display of a formal p-divisible group." In: *Cohomologies p-adiques et applications arithmétiques (I)*. Paris: Société Mathématique de France, 2002, pp. 127–248. ISBN: 2-85629-115-5/pbk.

Wissenschaftlicher Werdegang

2010	Abitur, Gymnasium Antonianum Geseke
2010–2011	Zivildienst
2011–2014	B. Sc. in Mathematik (Nebenfach Informatik), Universität Paderborn
2014–2016	M. Sc. in Mathematik, Universität Paderborn
2016–2020	wissenschaftlicher Mitarbeiter, Fachbereich Mathematik der Technischen Universität Darmstadt
2020	Promotion in Mathematik